

THE PROBLEM OF SCALES: RENORMALIZATION AND ALL THAT

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Abstract

I explain the methods that are used in field theory for problems involving typical momenta on two or more widely disparate scales. The principal topics are: (a) renormalization, which treats the problem of taking an ultra-violet cut-off to infinity, (b) the renormalization group, which is used to relate phenomena on different scales, (c) the operator product expansion, which shows how to obtain the asymptotics of amplitudes when some of its external momenta approach infinity.

1 Introduction

There is a common principle in physics, and indeed in science in general: One should neglect negligible variables. For example, a meteorologist does not have to take in account the microscopic molecular structure of air. A chemist only needs to know the masses and electric charges of atomic nuclei, but not their detailed properties as given by QCD. At the other end of the scale, a particle physicist discussing collisions at current energies need not take gravitation into account.

Without this principle, one would in general have great difficulty in making the approximations that make theoretical calculations tractable. When discussing phenomena on one scale of distance, one has no need to treat phenomena on totally different scales, at least not in any great detail.

Unfortunately, the principle, in its simplest form, does not apply to quantum field theories. Consider, for example, the large p^2 limit of the propagator for a scalar field in a renormalizable theory. At lowest order we have

$$\frac{i}{p^2 - m^2} \simeq \frac{i}{p^2} \text{ when } |p^2|/m^2 \gg \infty.$$

This suggests that the large p^2 asymptote of the full propagator can be obtained simply by setting $m = 0$, and hence that by dimensional analysis the propagator

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is proportional to $1/p^2$:

$$S(p^2, m^2) \simeq G(p^2, 0) = \frac{\text{const}}{p^2}.$$

As is well known, this is false, since higher order corrections to the propagator are polynomial in $\ln p^2$ at large p^2 . This anomalous scaling behavior is directly associated with the need to renormalize ultra-violet divergences in the theory.

So we find ourselves with a set of related topics that form the subject of this set of lectures:

- Ultra-violet divergences and their renormalization.
- Anomalous scaling in high momentum limits; the renormalization group.
- Asymptotic behavior as some momenta get large; the operator product expansion (OPE).
- How these ideas, after being developed in a simple theory (ϕ^4 theory), apply to the standard model and other field theories.

The concepts as expounded in these lectures apply to situations that are formally characterized as “Euclidean”. Sterman’s lectures will build on them to treat problems of physical interest that are directly formulated in Minkowski space.

I have been quite sparing of references, except for statements of properties that I do not think are well-known. I apologize to authors whose work I have not recognized in the bibliography. The reader shown consult the bibliography for more detailed treatments, and for further references.

2 Basic ideas of renormalization theory

I will now summarize the basic ideas of renormalization theory, first by expounding a well-known example from classical physics and then by reviewing how renormalization works at the one-loop order in simple examples.

2.1 Renormalization in classical physics

Let m_0 be the bare mass of an electron, i.e., the mass of the electron itself. Then the total energy of the electron and of its electric field down to radius a is given by

$$\begin{aligned} mc^2 &= m_0 c^2 + \int_{|\mathbf{r}|>a} d^3\mathbf{r} \frac{e^2}{32\pi^2\epsilon_0 r^4} \\ &= m_0 c^2 + \frac{e^2}{8\pi\epsilon_0 a}. \end{aligned} \tag{1}$$

The resulting linear divergence as $a \rightarrow 0$ is a consequence of the assumption of a point electron and the of special relativity, and was well-known in the early part of this century.

In a consistent classical theory one would have to introduce a structure for the electron and some non-electromagnetic forces to hold the electric charge together. Then

$$\text{observed mass of electron} = \text{bare mass} + \frac{\text{energy in e-m and non-e-m fields}}{c^2}. \quad (2)$$

The effect of the energy of the field on the mass of the electron is very large: if we set $a = 10^{-15}\text{m}$, then the electromagnetic energy is about 0.7 MeV, rather larger than the total energy. We now know that the electron is point-like to at least several orders of magnitude smaller in distance, so the classical electromagnetic energy is much larger than the total energy. Of course, the calculation above is purely classical, and it must be modified by quantum field theoretical effects. But it illustrates some important principles.

One is that a basic parameter, such as a mass, does not have to equal the observed quantity of the same name.

Another is that it is convenient to write all calculations in terms of the measured mass m . and then we write the bare mass as the observed mass plus a counterterm:

$$m_0 = m + \delta m, \quad (3)$$

with the counterterm being adjusted to cancel the energy of the fields that surround an isolated electron. This procedure we call renormalization of the mass. At this point we may take the radius $a \rightarrow 0$, and we will no longer encounter divergences in directly measurable quantities.

From a practical point of view a very large finite value for the self energy of an electron is almost as bad as an infinity. In either case, the field energy is a large effect, and we must express calculations in terms of measured quantities.

2.2 Divergences in quantum field theory

All the above issues, and in particular the divergences, are even more pervasive in quantum field theory; there are infinities everywhere, but they appear to be rather more abstract in nature.

To explain the principles of how these divergences arise and of how they are cancelled by renormalization counterterms, I will use the simplest possible theory, ϕ^4 in four space-time dimensions. Its Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4. \quad (4)$$

The Feynman rules for the theory are given by a propagator $i/(p^2 - m^2 + i\epsilon)$ and a vertex $-i\lambda$. Although we will wish to treat the theory in four dimensions, it will be convenient to ask what happens if the space-time dimensionality had some other value n . Dimensional analysis will play an important role in our discussions, so we note now that the field and coupling have energy dimensions $[\phi] = E^{n/2-1}$ and $[\lambda] = E^{4-n}$. These follow from the fact that the units of \mathcal{L} are those of an energy density, since we use natural units, i.e., with $\hbar = c = \epsilon_0 = 1$.

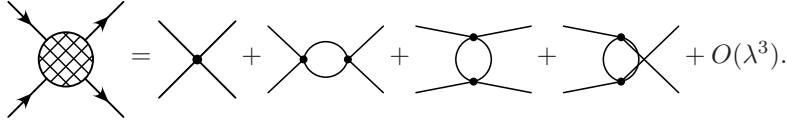


Figure 1: Connected and amputated 4-point Green function in ϕ^4 theory.

All the same principles will apply to more general theories, and all theories in four-dimensional space-time have divergences, so that we cannot evade the issues by trying another theory. Indeed, it can readily be seen from other lectures in this book that the topics I will discuss are essential in treatments of the standard model and of its many proposed extensions.

The graphs up to one-loop order for the connected and amputated 4-point Green functions are shown in Fig. 1. Each of the one-loop graphs is the same except for a permutation of the external lines, so it will suffice to examine the integral

$$I(p^2) = \frac{(-i\lambda)^2}{2} \int \frac{d^n k}{(2\pi)^n} \frac{i^2}{(k^2 - m^2 + i\epsilon) [(p - k)^2 - m^2 + i\epsilon]}. \quad (5)$$

Possible divergences at the propagator poles are avoided by contour deformations, according to the specified $i\epsilon$ prescriptions. The only remaining divergence is an ‘ultra-violet’ divergence as $k \rightarrow \infty$. At the physical space-time dimension, $n = 4$, this is a logarithmic divergence.

In general, we define the *degree of divergence*, Δ , for a graph by counting powers of k at large k . For Eq. (5) this is $\Delta = n - 4$. Evidently, the graph is convergent if $\Delta < 0$, i.e., if $n < 4$, and it is convergent if $\Delta \geq 0$, i.e., if $n \geq 4$.

A convenient (and in fact general) way of seeing how the divergence arises is to observe that the degree of divergence is equal to the energy dimension of the integral over k . Since the one-loop graph has the same dimension as the lowest order graph, whose value is $-i\lambda$, we have

$$\dim(\text{tree graph}) = \dim \lambda = \dim(\text{loop graph}) = \dim \lambda^2 + \Delta, \quad (6)$$

so that

$$\Delta = -\dim \lambda = n - 4. \quad (7)$$

Thus the existence of an ultra-violet divergence is directly associated with the fact of having a coupling of zero or negative dimension.

2.3 Interpretation of divergence

Since we have obtained an infinite result for the calculation, the theory is no good, at least as formulated naïvely. (The fault is not just that of perturbation theory, although that is not apparent at this stage.) Ultra-violet divergences are typical properties of relativistic quantum field theories and are particularly

associated with the need for a continuous manifold for space-time and for local interactions in a relativistic theory.

To remove the divergences, we first impose a “cut-off” on the theory, to regulate the ultra-violet divergences. This typically removes some desirable properties: the use of a lattice space-time, for example, wrecks Poincaré invariance. Then we adjust the parameters (λ, m) as functions of the cut-off, so that finite limits are obtained for observable quantities when we finally remove the cut-off.

This is an obvious generalization of the procedure for the classical electron. Our formulation is non-perturbative, even though typical applications are within perturbation theory. If one does not like even the implicit divergences, then one need not take the limit of completely removing the cut-off, but only go far enough that the effect of the cut-off, after renormalization, is negligible in actual experiments.

2.4 Renormalization of one-loop graph

Suppose that we have a lattice for space or space-time, with lattice spacing a . Then the free propagator $S(k, m; a)$ approaches its continuum value $i/(k^2 - m^2 + i\epsilon)$ when $|k^\mu| \ll 1/a$, while it is zero if $|k^\mu| > O(1/a)$. The regulated graph Eq. (5) can be written

$$\begin{aligned} I(p^2) &= \frac{(-i\lambda)^2}{32\pi^4} \int d^4k S(k, m; a) S(p - k, m; a) \\ &\simeq \frac{(-i\lambda)^2}{32\pi^4} \left\{ \int_{|k|< k_0} d^4k \frac{i^2}{(k^2 - m^2 + i\epsilon)[(p - k)^2 - m^2 + i\epsilon]} \right. \\ &\quad \left. + \int_{|k|> k_0} d^4k S(k, 0; a)^2 \right\}. \end{aligned} \quad (8)$$

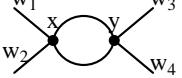
Here we separate small and large momenta by a parameter k_0 that obeys $p, m \ll k_0 \ll 1/a$. Since the divergent part of the integral does not depend on the external momentum p , we can cancel the divergence by a “counterterm” that corresponds to a renormalization of the ϕ^4 interaction vertex:

$$\begin{aligned} & \left[\text{Diagram } 1 + \text{Diagram } 2 \right] + \text{Diagram } 3 + \text{Diagram } 4 + \text{Diagram } 5 + O(\lambda^3). \\ &= -i\lambda + i\lambda^2 \{ \text{Complicated piece independent of } a + C(a) + O(|pa|) \} \\ &\quad -i\Delta\lambda + O(\lambda^3), \end{aligned} \quad (9)$$

where $C(a)$ diverges as $a \rightarrow 0$. If we set $\delta\lambda = C(a) + \text{finite}$, we will get a finite result in the limit $a \rightarrow 0$ with the renormalized coupling λ fixed.

2.5 Coordinate space

(See [1] for the material in this section.) The same loop graph as before can be written in coordinate space as



$$= \frac{(-i\lambda)^2}{2} \int d^4x d^4y [i\Delta_F(x-y)]^2 f(x, y, w_1, w_2, w_3, w_4), \quad (10)$$

where the propagator is $i\Delta_F$, and the function f is the product of the four external propagators. Since

$$i\Delta_F(x-y) \sim \frac{-1}{4\pi^2(x-y)^2} \quad (11)$$

as $x \rightarrow y$, there is a logarithmic divergence at $x = y$, and we can cancel the divergence by a counterterm proportional to $\delta^{(4)}(x-y)$. Thus the value of the graph plus counterterm has the form

$$\lim_{a \rightarrow 0} \frac{(-i\lambda)^2}{2} \int d^4x d^4y \left\{ [i\Delta_F(x-y)]^2 + C(a)\delta^{(4)}(x-y) \right\} f(x, y, \underline{w}), \quad (12)$$

where I have indicated the limit that an ultra-violet regulator is removed.

2.6 Modified Lagrangian

Our discussions indicate that we should attempt to cancel the UV divergences by renormalizing all the parameters in the Lagrangian. Thus we replace the original Lagrangian Eq. (4) by

$$\mathcal{L} = \frac{1}{2}Z(\partial\phi)^2 - \frac{1}{2}m_B^2\phi^2 - \frac{1}{4!}\lambda_B\phi^4, \quad (13)$$

where Z , m_B^2 and λ_B are singular as the lattice spacing is taken to zero. By changing variables according to $\phi_0 = \sqrt{Z}\phi$, $m_0^2 = m_B^2/Z$, and $\lambda_0 = \lambda_B/Z^2$, we find

$$\mathcal{L} = \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{1}{4!}\lambda_0\phi_0^4, \quad (14)$$

where the “kinetic energy term” has a unit coefficient. We call ϕ_0 , m_0 and λ_0 the bare field, bare mass and bare coupling.

For doing perturbation theory, we separate \mathcal{L} into three pieces:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \\ & - \frac{1}{4!}\lambda\phi^4 \\ & \frac{1}{2}\delta Z(\partial\phi)^2 - \frac{1}{2}\delta m^2\phi^2 - \frac{1}{4!}\delta\lambda\phi^4. \end{aligned} \quad (15)$$

Here we have chosen arbitrarily a renormalized mass m and a renormalized coupling λ that are to be held fixed as the UV regulator is removed. The first line, the “free Lagrangian”, is used to make the propagators in the Feynman rules, and the remaining terms are used to make the interactions, to be treated in perturbation theory. The second line gives the “basic interaction” and the third line, the “counterterm Lagrangian” is used to cancel divergences in graphs containing basic interaction vertices. The counterterms are expressed in terms of the renormalized parameters, and the renormalized coupling λ is used as the expansion parameter of perturbation theory.

It is to be emphasized that the three formulae (13–15), refer to the same Lagrangian, which is just written differently. Some people talk about a different bare and renormalized Lagrangian, but this is not the case in the way I have formulated renormalization. The one Lagrangian that is different, Eq. (4) is not actually used for physical calculations; indeed it gives unambiguously divergent Green functions. I will sometimes refer to the Eq. (14) as the “bare Lagrangian”, but what I will actually mean is the “Lagrangian written in terms of bare fields”.

2.7 Dimensional regularization

With suitable adjustments of the finite parts of the counterterms, the same results are obtained for renormalized Green functions (i.e., Green functions of the renormalized field ϕ) no matter which UV regulator is chosen. Hence we can use the regulator that is most convenient for doing calculations.

For many purposes, dimensional regularization is very convenient. Here, we treat the dimension n of space-time as a continuous parameter, $n = 4 - \epsilon$. We remove the regulator by taking $\epsilon \rightarrow 0$. It can be shown (e.g., [1]) that the integrals over n dimensional vectors can be consistently defined, so that dimensional regularization is consistent, at the level of perturbation theory.

The one complication is that the coupling has mass dimension ϵ . To make calculations manifestly dimensionally correct, we introduce an arbitrary parameter with the dimensions of mass, the “unit of mass” μ and write the bare coupling as

$$\lambda_0 = \mu^\epsilon \lambda + \text{counterterms}, \quad (16)$$

where now the renormalized coupling is dimensionless for all ϵ .¹ The one-loop integral Eq. (5) is now

$$I(p^2) = \frac{\lambda^2 \mu^{2\epsilon}}{2(2\pi)^{4-\epsilon}} \int d^{4-\epsilon} k \frac{1}{(k^2 - m^2 + i\epsilon) [(p - k)^2 - m^2 + i\epsilon]}. \quad (17)$$

Calculation and renormalization of Feynman graphs become easy with dimensional regularization. For example, in Eq. (17), we combine the denominators

¹The reader will probably notice that there is a considerable arbitrariness in the conversion of the Lagrangian in its bare form Eq. (14) to Feynman rules. A change of the value of μ changes the split of the Lagrangian between free, basic and counterterm parts. As we will see in our discussion of the renormalization group, Sect. 4, such changes can be compensated by changes in the values of the renormalized parameters. Instead of being a nuisance, they give useful methods of optimizing the information obtained by finite order calculations.

nators by the Feynman parameter method Eq. (114), shift the integral over k and then perform a Wick rotation: $k^0 = i\omega$ to obtain a spherically symmetric integral over a Euclidean momentum:

$$I(p^2) = \frac{i\lambda^2\mu^{2\epsilon}}{2(2\pi)^{4-\epsilon}} \int_0^1 dx \int d_E^{4-\epsilon} k \frac{1}{[m^2 - p^2 x(1-x) + k_E^2]^2}, \quad (18)$$

where $d_E^{4-\epsilon} k = d\omega d^{3-\epsilon} \mathbf{k}$, and $k_E^2 = \omega^2 + \mathbf{k}^2 = -k^2$. It can readily be shown that a spherically symmetric integral can be reduced to a radial integral Eq. (115), and hence we get

$$\begin{aligned} I(p^2) &= \frac{i\lambda^2\mu^\epsilon}{32\pi^2} \Gamma(\epsilon/2) \int_0^1 dx \left[\frac{m^2 - p^2 x(1-x)}{4\pi\mu^2} \right]^{-\epsilon/2} \\ &= \frac{i\lambda^2\mu^\epsilon}{32\pi^2} \left\{ \frac{2}{\epsilon} - \int_0^1 dx \ln \left[\frac{m^2 - p^2 x(1-x)}{4\pi\mu^2} \right] - \gamma_E + O(\epsilon) \right\}, \end{aligned} \quad (19)$$

where $\gamma_E = 0.5772\dots$ is Euler's constant. In obtaining the first line of this equation, we have in effect proved Eq. (116).

We can now choose the counterterm

$$\delta\lambda = \frac{3\lambda^2\mu^\epsilon}{16\pi^2\epsilon} + O(\lambda^3) \quad (20)$$

to cancel the divergence in the three one-loop graphs. Then after adding the graphs and the counterterms we set $\epsilon = 0$ and obtain the renormalized amputated four-point Green function:

$$-i\lambda - \frac{i\lambda^2}{32\pi^2} \sum_{p^2=s,t,u} \left\{ \int_0^1 dx \ln \left[\frac{m^2 - p^2 x(1-x)}{4\pi\mu^2} \right] + \gamma_E \right\} + O(\lambda^3). \quad (21)$$

Given our choice of the counterterm which is just a pure pole at $\epsilon = 0$, we call λ the renormalized coupling in the minimal subtraction scheme.

2.8 Renormalizability

We have shown that particular UV divergences in quantum field theory can be removed by renormalization of the parameters of the lagrangian. The question now arises as to which theories can have all their divergences renormalized away, a question which we will address in Sects. 3 and 6. It is convenient to make the following definitions:

1. A quantum field theory is *renormalizable* if it can be made finite as the UV regulator is removed by suitably adjusting coefficients in \mathcal{L} .
2. A quantum field theory is *non-renormalizable* if it is not renormalizable. (To be strictly correct, I should say that in practice one says that a theory is non-renormalizable if it merely has not been shown to be renormalizable. For example, proofs within perturbation theory do not automatically imply anything about non-perturbative properties of a theory.)

Typically proofs are in perturbation theory, to all orders.

2.9 Divergent (one-loop) graphs in $(\phi^4)_4$

An important element of proofs of renormalizability is the concept of the “degree of divergence”, $\Delta(\Gamma)$, of a graph Γ . As I have already said, this is defined by counting the number of powers of loop momenta in the graph when these loop momenta get large. A one-loop graph Γ is UV divergent if $\Delta(\Gamma) \geq 0$ and UV convergent if $\Delta(\Gamma) < 0$. One should only include the one-particle-irreducible (1PI) part of the graph in this counting. The same concept will be applied to higher-order graphs in Sect. 3.

In ϕ^4 theory in 4 dimensions, one can easily show that, for a general 1PI graph

$$\Delta = 4 - \text{number of external lines.} \quad (22)$$

Hence the 4-point function is logarithmically divergent, and the only other graphs that have a divergence are the self-energy graphs, with $\Delta = 2$. By differentiating self-energy graphs three times with respect to the external momenta we obtain convergent integrals, with $\Delta = -1$. The necessary counterterms are therefore quadratic in external momenta; given Lorentz invariance, they have the form $i\delta Z p^2 - i\delta m^2$. Corresponding vertices are obtained from “wavefunction” and mass counterterms in the Lagrangian: $\frac{1}{2}\delta Z(\partial\phi)^2 - \frac{1}{2}\delta m^2\phi^2$, as in Eq. (15). It now follows that ϕ^4 theory is renormalizable at the one-loop level. (In fact the topology of the one-loop self-energy graph makes it have no momentum dependence, so that δZ starts at the next order, λ^2 .)

2.10 Summary

1. A parameter in the Lagrangian or Hamiltonian need not equal a measured quantity of the same name.
2. We can use simple power counting methods to determine a degree of divergence for Feynman graphs, and hence readily determine which are divergent.
3. We can cancel divergences by renormalization – adjustment of the values of parameters in the Lagrangian compared to the values one first thought of.
4. Renormalization can be implemented in perturbation calculations by a counterterm technique.
5. We have reviewed some practical methods for calculations, including dimensional regularization.

3 All-orders Renormalization

In this section I will summarize how one generalizes the one-loop results of the previous section to all orders of perturbation theory. This subject has a reputation for being very abstruse. To make it as comprehensible as possible, proper organization is vital. The distribution-theoretic methods developed by Tkachov and collaborators [7] are very useful, and my presentation will use them.

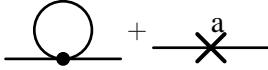


Figure 2: Renormalized self energy graph.

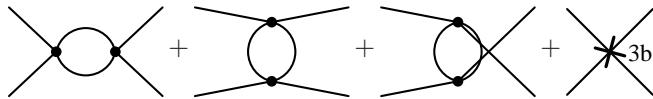


Figure 3: Renormalized 1-loop four-point function.

I will also work in coordinate space. This makes the demonstrations more intuitive than the corresponding ones in momentum-space. Of course a theorem about Feynman graphs in coordinate space immediately implies the corresponding theorem in momentum space, which is where practical calculations are normally done.

My starting point is the decomposition Eq. (15) of the Lagrangian into a free Lagrangian, a basic interaction term and counterterms. Each counterterm will be treated as a sum of terms, one for each “overall divergent” graph generated by the basic interaction.

We organize perturbation theory in powers of the renormalized coupling by first writing the graphs of some given order that are obtained by using only the basic interaction to form vertices; these graphs have divergences. Then to each such basic graph we add all possible counterterm graphs; these are obtained by replacing particular subgraphs by the corresponding counterterm vertices in the counterterm Lagrangian. The total of each basic graph and its associated counterterm graphs will be finite if the renormalization procedure works.

Examples of this organization are shown in Figs. 2 and 3 for the ϕ^4 theory. In these figures we label the counterterms for each divergent basic graph by a letter. For example, in Fig. 2, “*a*” represents the one-loop mass counterterm

$$-i\delta m_a^2 = -i\frac{\lambda m^2}{16\pi^2\epsilon}, \quad (23)$$

which can be calculated by the same methods we used in Sect. 2.7. In Fig. 3, the counterterm “3*b*” is the one in Eq. (20); the “3” in “3*b*” is an indication that there are three basic graphs, each with a numerically identical counterterm.

3.1 Dimensionally regularization and coordinate space

In Sect. 2.5, I summarized how to renormalize the one-loop four-point function in coordinate space. Now I will give some details that are needed to be able to do these calculations in detail, with dimensional regularization. The free

propagator and its asymptotic short-distance behavior are:

$$\begin{aligned} i\Delta_F(x-y) &= \frac{m^{n/2-1}}{(2\pi)^{n/2}} [-(x-y)^2]^{1/2-n/4} K_{n/2-1} \left(\sqrt{-m^2(x-y)^2} \right) \\ &\sim \frac{\Gamma(n/2-1)}{4\pi^{n/2} (-x-y)^{n/2-1}} \quad \text{as } x \rightarrow y. \end{aligned} \quad (24)$$

Here $K_{n/2-1}$ is a modified Bessel function of order $n/2 - 1$. Then the dimensionally regularized value of one of the one-loop graphs, plus its counterterm, is

$$\frac{(-i\lambda)^2 \mu^\epsilon}{2} \int d^n x d^n y f(x, y, \underline{w}) \left\{ \mu^\epsilon [i\Delta_F(x-y)]^2 + \frac{i}{8\pi^2 \epsilon} \delta^{(n)}(x-y) \right\}. \quad (25)$$

It should be noted that the limit as $\epsilon \rightarrow 0$ of Eq. (25) can be considered as the integral of a test function $f(x, y, \underline{w})$ with a generalized function. The generalized function is the factor in braces $\{\dots\}$, and can be considered a generalization of the generalized function $(1/x)_+$ in one dimension.

3.2 List of results

The mathematical problem to be solved is to understand the asymptotics of the integration of multidimensional integrals in momentum space as some or all of the integration variables go to infinity. Equivalently, it is to understand the short-distance asymptotics of the corresponding coordinate-space integrals.

Our results will be

1. To associate a counterterm with each 1PI graph that has an overall divergence.
2. To show that the result is finite when we add to a basic graph the counterterms for its divergent subgraphs.
3. To obtain the power-counting techniques that determine the necessary counterterms.
4. To implement the counterterms as extra terms in the Lagrangian.

3.3 The Bogoliubov R -operation

We define the renormalized value of a basic graph to be the sum of the graph and all the counterterms for its divergences. The operation of constructing the renormalized value $R(\Gamma)$ of a basic graph Γ was formalized by Bogoliubov, hence the name ‘‘Bogoliubov R -operation’’. An N -point Green function is then to be considered as $\sum_\Gamma R(\Gamma)$, the sum being over all basic graphs for the Green function.

Consider first a graph consisting of some one-loop 1PI graphs connected together, as in Fig. 4. The one-loop counterterm vertices from \mathcal{L} imply the

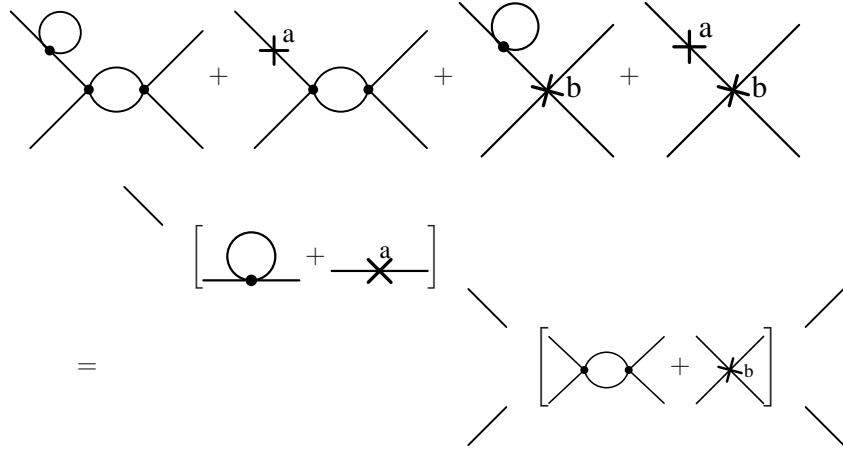


Figure 4: An example of a renormalized two-loop graph.

three counterterm graphs in which one or both the divergent one-loop subgraphs is replaced by its counterterm. As shown in the lower part of the figure, the counterterms may be organized to give a product form.

This example is evidently a case of a general set of results:

- UV divergences are confined to 1PI (sub)graphs.
- Each divergent 1PI subgraph γ “owns” a piece of counterterm $C(\gamma)$.
- To see finiteness easily, we should keep a graph and its counterterm together.

Moreover we can easily generalize Fig. 4 to the following theorem:

Suppose a basic graph Γ is a product of lines outside loops and a set of 1PI graphs:

$$\Gamma = \prod \text{propagators outside loops} \times \prod \text{1PI } \gamma. \quad (26)$$

Then its renormalized value is

$$R(\Gamma) = \prod \text{propagators outside loops} \times \prod R(\gamma). \quad (27)$$

3.3.1 Simple non-trivial case

The basic graph in Fig. 5 is

$$\frac{i\lambda^3\mu^{3\epsilon}}{4} \int d^n x d^n y d^n z f(x, y, z, \underline{w}) [i\Delta_F(x - y)]^2 [i\Delta_F(y - z)]^2, \quad (28)$$

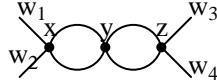


Figure 5: A two-loop graph.

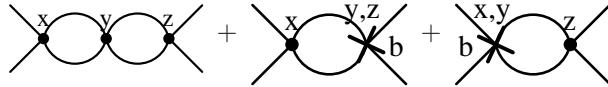


Figure 6: The two-loop graph of Fig. 5 plus counterterms for its subdivergences.

where $f(x, y, z, \underline{w})$ is the product of the four external propagators $i\Delta_F(w_1 - x)i\Delta_F(w_2 - x)i\Delta_F(w_3 - z)i\Delta_F(w_4 - z)$. The singularities of the integrand are on the hyperplanes $x = y$ and $z = y$ and at their intersection $x = y = z$. We ignore for this purpose the *integrable* singularities of the four external propagators.

The Feynman rules tell us that we have counterterms for the subdivergences, i.e., for the singularities on each of the hyperplanes $x = y$ and $y = z$. After adding these counterterms, as in Fig. 6, we get

$$\begin{aligned} & \frac{i\lambda^3\mu^\epsilon}{4} \int d^n x d^n y d^n z f(x, y, z, \underline{w}) \times \\ & \left\{ \mu^\epsilon [i\Delta_F(x - y)]^2 \mu^\epsilon [i\Delta_F(y - z)]^2 \right. \\ & \left. + \mu^\epsilon [i\Delta_F(x - y)]^2 iC(\epsilon) \delta^{(n)}(y - z) + iC(\epsilon) \delta^{(n)}(x - y) \mu^\epsilon [i\Delta_F(y - z)]^2 \right\}, \end{aligned} \quad (29)$$

where $C(\epsilon) = 1/(8\pi^2\epsilon)$ is the same counterterm coefficient as in Eq. (25). When we take $\epsilon \rightarrow 0$, the integral is finite everywhere except at the intersection of the singular surfaces of the subdivergences, i.e., at $x = y = z$. Thus we have subtracted the subdivergences. Therefore we make the following definitions:

- A *subdivergence* is where the positions of a strict subset of vertices approach each other and give a divergence.
- An *overall divergence* occurs where the positions of all the vertices approach each other and give a divergence.

For example, Fig. 5 has 2 subdivergences, at $x = y$ and $y = z$ and has an overall divergence at $x = y = z$.

A general method of finding the overall divergence is illustrated by this graph. Fix y , and consider the integral over $x - y$ and $z - y$; this will be sufficient to find the divergences. Now express the integration variables in terms of a radial variable λ and some angular integrals over a $7 - 2\epsilon$ surface surrounding the point $z - y = 0 = x - y$, Fig. 7. We write

$$(x - y, z - y) = \lambda(\hat{x} - \hat{y}, \hat{z} - \hat{y}), \quad (30)$$

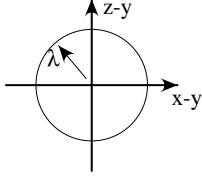


Figure 7: Definition of radial and angular variables.



Figure 8: Fig. 5 plus all its counterterms.

so that

$$\int d^n x \int d^n z = \int d\lambda \lambda^{2n-1} \int d^{2n-1}(\hat{x} - \hat{y}, \hat{z} - \hat{y}). \quad (31)$$

The integral over the hatted variables gives no divergence, because of the subtractions in Eq. (29), and simple power counting then shows that there is a logarithmic divergence at $\lambda = 0$.²

Since the divergence is logarithmic, it may be subtracted by a delta-function counterterm, as in Fig. 8, so that the renormalized value of the graph is

$$\frac{i\lambda^3 \mu^\epsilon}{4} \int d^n x d^n y d^n z f(x, y, z, \underline{w}) \times \\ \left\{ \text{Same as in Eq. (29)} + C_f(\epsilon) \delta^{(n)}(x - y) \delta^{(n)}(z - y) \right\}, \quad (32)$$

where calculation gives

$$C_f(\epsilon) = \frac{-1}{(8\pi^2 \epsilon)^2}. \quad (33)$$

3.4 The R -operation

The structure of the previous example is quite general.³ The general procedure for obtaining the renormalized value $R(G)$ of a Feynman graph G is summarized in the following formula:

$$R(G) = G + \sum_{\gamma_1, \dots, \gamma_n} G \Big|_{\gamma_i \rightarrow C(\gamma_i)}. \quad (34)$$

² The subtracted logarithmic singularities of the integrand at $x = y$ and $z = y$ include a term proportional to $\ln \lambda$ when the positions are scaled by a factor λ . This logarithm does not affect the counting of powers, of course.

³ It in fact gives the simplest example of what is called an “overlapping divergence”. Such divergences were considered a difficult problem in the early days of renormalization theory, but with proper organization they are no more difficult to handle than non-overlapping divergences in multi-loop graphs.

The sum is over all sets of non-intersecting 1PI subgraphs of G , and each of the 1PI subgraphs γ_i is to be replaced by its counterterm $C(\gamma_i)$. The counterterm $C(\gamma)$ of a 1PI graph γ is constructed in the form

$$C(\gamma) = -T(\gamma + \text{Counterterms for subdivergences}). \quad (35)$$

Here T is an operation that defines the renormalization scheme. For example, in minimal subtraction we define

$$T(\Gamma) = \text{pole part at } \epsilon = 0 \text{ of } \Gamma. \quad (36)$$

In Eq. (35), we can formalize the term inside square brackets as follows:

$$\begin{aligned} \bar{R}(\gamma) &\equiv \gamma + \text{Counterterms for subdivergences} \\ &= \gamma + \sum'_{\gamma_1, \dots, \gamma_n} \gamma \Big|_{\gamma_i \rightarrow C(\gamma_i)}, \end{aligned} \quad (37)$$

where the prime ($'$) on the \sum' denotes that we sum over all sets of non-intersecting 1PI subgraphs except for the case that there is a single γ_i equal to the whole graph (i.e., $\gamma_1 = \gamma$). Thus

$$R(\gamma) = \bar{R}(\gamma) + C(\gamma). \quad (38)$$

The formulae Eq. (34)–(37) form a recursive construction of the renormalization of an arbitrary graph. The recursion starts on one-loop graphs, since they have no subdivergences, i.e., $C(\gamma) = -T(\gamma)$ for a one-loop 1PI graph.

The important non-trivial analytic result, an example of which we saw in the previous subsection, is that power counting in its naïvest form determines what counterterms are needed for a given 1PI (sub)graph. The counterterm $C(\gamma)$ is a polynomial of degree equal to the overall degree of divergence of γ . Thus the naïve degree of divergence equals the actual degree of divergence, but only after subtraction of subdivergences.

Once the counterterms have been constructed, we may implement them as counterterms in the Lagrangian. Thus for the ϕ^4 theory

$$-i\delta\lambda = \sum C(\text{1PI graph for 4-point function}), \quad (39)$$

and

$$i\delta Zp^2 - i\delta m^2 = \sum C(\text{1PI self-energy graphs}). \quad (40)$$

Another example of 2-loop renormalization is shown in Fig. 9. The symmetry factors work out such that two copies of the one-loop counterterm are needed for the subdivergence.

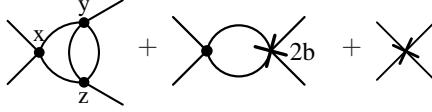


Figure 9: Another two-loop graph.

3.5 Translation to momentum space

We may translate the above results to momentum space. For example, the 1PI part of the two-loop graph of Fig. 5 in momentum space is

$$\frac{i\lambda^3\mu^{3\epsilon}}{4(2\pi)^{8-2\epsilon}} \int d^n k d^n l \frac{1}{(k^2 - m^2)} \frac{1}{[(p-k)^2 - m^2]} \frac{1}{(l^2 - m^2)} \frac{1}{[(p-l)^2 - m^2]}. \quad (41)$$

It has subdivergences when $k \rightarrow \infty$ with l fixed, and when $l \rightarrow \infty$ with k fixed, and it has an overall divergence when both $k, l \rightarrow \infty$. The interplay between the subdivergences and the overall divergence is harder to conceptualize when they are at infinity, as they are in momentum space, than when they are localized on lower dimensional surfaces and at points, as they are in coordinate space.

But the calculations are quite easy, particularly for Fig. 5. The basic graph is just the square of a simple integral:

$$\begin{aligned} & -i\lambda^3\mu^\epsilon I(p^2, m^2, \epsilon)^2 \\ &= -i\lambda^3\mu^\epsilon \left\{ \frac{A}{\epsilon} - \frac{A}{2} \int_0^1 dx \ln \left[\frac{m^2 - p^2 x(1-x)}{\mu^2} \right] + \text{constant} + O(\epsilon) \right\}^2, \end{aligned} \quad (42)$$

where $A = 1/(16\pi^2)$. This integral has a double pole ($\propto 1/\epsilon^2$) at $\epsilon = 0$, which is evidently a characteristic of two loop graphs with a divergent subgraph inside the overall divergence. But the single pole ($\propto 1/\epsilon$) has a non-polynomial coefficient. Hence, before we subtract subdivergences we cannot renormalize the graph.

After subtraction of subdivergences, the only divergences are independent of the external momentum, as befits a logarithmic divergence:

$$\begin{aligned} \bar{R}(\text{Fig. 9}) &= -i\lambda^3\mu^\epsilon (I^2 - 2I \times \text{pole}(I)) \\ &= -i\lambda^3\mu^\epsilon [(I - \text{pole}(I))^2 - \text{pole}(I)^2] \\ &= -i\lambda^3\mu^\epsilon \left(\text{finite} - \frac{A^2}{\epsilon^2} \right). \end{aligned} \quad (43)$$

Hence the graph can be renormalized by a coupling counterterm, the same one as we calculated by coordinate space methods.

Note that at large p^2 , the finite part behaves like $\ln^2 p^2$ plus smaller terms, whereas a one-loop graph only has one logarithm. The double logarithm is characteristic of two-loop graphs with subdivergences.

3.6 Summary

1. Suppose we have added to the Lagrangian a counterterm $C(\gamma)$ for a particular graph γ . Then whenever that graph occurs as a subgraph of a bigger graph Γ , the Feynman rules require that we also have a graph in which γ is replaced by its counterterm. Thus we must cancel the subdivergences of a graph before asking how to cancel its overall divergence.
2. After subtraction of subdivergences, the only remaining divergence in a graph Γ is its overall divergence. Its strength is given by the overall degree of divergence $\Delta(\Gamma)$, which is determined by simple power counting.
3. The overall divergence is cancelled by a new contribution to the counterterm Lagrangian.
4. As already implied, counterterms to 1PI graphs are implemented as counterterms in the Lagrangian.
5. Power counting for ϕ^4 in 4 space-time dimensions shows, as we saw earlier at Eq. (22), that the only counterterms needed are those corresponding to terms in the original Lagrangian. That is, the theory is renormalizable to all orders of perturbation theory.

4 Renormalization Group

We have seen in examples how one cancels the divergences in Feynman graphs by adding divergent counterterms to the Lagrangian. Although the divergent parts of the counterterms are determined by the requirement that we obtain finite renormalized Green functions, the finite parts are not fixed. So we have an apparent freedom to make arbitrary changes in the results of calculations by choosing the finite parts of the counterterms. A prescription for choosing the finite parts is called a renormalization prescription or a renormalization scheme.

At first sight, the dependence of predictions on the choice of renormalization scheme appears to remove the predictive power of a theory, and hence to be undesirable physically. In fact, as we will see, a change of renormalization scheme can be completely compensated by changes in the numerical values of the finite renormalized parameters of the theory, which remains invariant. The *renormalization group* is the formulation of this invariance. A change of renormalization scheme is, in essence, just a kind of non-linear change of the units for the renormalized parameters. Another way of saying this is that it amounts to a change of the splitting of the Lagrangian into basic and counterterm parts Eq. (15).

This suggests that, far from being a nuisance, renormalization scheme dependence gives a powerful method for improving perturbation calculations. Consider the one loop calculation Eq. (21). At large p^2 , the coefficient of λ^2 has a logarithm of p^2 , which ruins the accuracy of low-order perturbation theory. Note that higher order calculations get an extra logarithm per loop — see Eqs. (42) and (43). One change of renormalization prescription is to change the value of the unit of mass μ ; then we can try to get rid of the large logarithms by a

choice of μ . This works in cases that all the momentum scales in a problem are comparable. In Sect. 5, we will see some of the techniques that can be used when there is more than one typical scale in a problem.

When one changes μ , one must compensate by changing the coupling, so one comes to the concept of the *running coupling*, or *effective coupling*, $\lambda(\mu)$. Perturbative calculations are valid if there are no large logarithms and if the effective coupling is small. One important result are the renormalization group equations (RGEs), which are differential equations that enable the running of the coupling, for example, to be computed from elementary perturbative calculations.

In this section, we will explain the above concepts.

4.1 Renormalization Schemes

The calculation of the four point function in ϕ^4 theory can be summarized as follows

$$\begin{aligned} G_4 &= -i\lambda & (44) \\ &\quad - \frac{i\lambda^2}{32\pi^2} \{ \text{Pole} + \text{Complicated finite term} + \text{Counterterm} \} + O(\lambda^3). \end{aligned}$$

Let us add and subtract a finite term $-ia\lambda^2/(32\pi^2)$, with a being an arbitrary finite number:

$$\begin{aligned} G_4 &= -i \left[\lambda - \frac{a\lambda^2}{32\pi^2} \right] \\ &\quad - \frac{i\lambda^2}{32\pi^2} \{ \text{Pole} + \text{Complicated finite term} + \text{Counterterm} + a \} \\ &\quad + O(\lambda^3). & (45) \end{aligned}$$

We have added an extra piece to the counterterm and compensated by changing the value of the coupling to

$$\lambda' = \lambda - \frac{a\lambda^2}{32\pi^2}. & (46)$$

We should, of course, change the coupling in the one-loop part of Eq. (45) from λ to λ' . At the level of accuracy we are working to, λ^2 , the change will be of the same as the higher order corrections that we have not yet calculated.

We make the following definitions:

A *renormalization scheme* (or *prescription*) is a rule for deciding the finite part of a counterterm. Equivalently, it is a rule for defining the meaning of the renormalized parameters of a theory.

Renormalization group invariance is the statement that when one changes the renormalization scheme, one keeps the same physics by suitable changes of λ , etc.

When the external momenta are large, in our ϕ^4 example, we find that

$$G_4 = -i\lambda - \frac{i\lambda^2}{32\pi^2} \left[\ln \frac{s}{\mu^2} + \ln \frac{t}{\mu^2} + \ln \frac{u}{\mu^2} + \text{constant} \right] + O(\lambda^3). \quad (47)$$

We can keep the coefficient of the one-loop term small by choosing μ , but only if the three invariants s , t , and u are comparable in size.

4.2 Particular renormalization schemes

Among the infinity of possible renormalization schemes, only a few are commonly used, and I list here some of the definitions

Physical Renormalization schemes A physical scheme is commonly used in QED. There one defines the renormalized mass of the electron to be exactly its physical value. The renormalized coupling e is defined so that the long range part of the electric field of an electron is the usual $e/(4\pi r^2)$ (in units with $\epsilon_0 = 1$). In terms of Green functions, the definition of the mass is that the pole in the electron propagator is at $p^2 = m^2$, i.e., that the self energy graphs are zero at $p^2 = m^2$. The definition of the coupling is that the 1PI $ee\gamma$ Green function is equal to its lowest order value when the electrons are on-shell and the momentum transfer to the photon is zero.

Momentum-space subtraction schemes Here, one defines the renormalized parameters to be equal to the values of appropriate Green functions at some value of the external momenta. At the chosen value of external momentum, all higher order corrections to the Green function are zero. A physical renormalization scheme is one case of this.

Minimal Subtraction Minimal subtraction (MS) is defined given a particular regularization scheme, normally dimensional regularization. Counterterms are defined to be just the singular terms needed to cancel the divergences. With dimensional regularization, the counterterms are just poles at $\epsilon = 0$. An extra mass scale, the unit of mass μ needs to be introduced.

A common modification is the modified minimal subtraction scheme, $\overline{\text{MS}}$, (with dimensional regularization). Here one redefines μ by a particular factor to cancel certain numerical terms that typically arise in a calculation using dimensional regularization.

Minimal subtraction can also be defined with other regulators. For example, with a lattice spacing a , one could define the counterterms in each order of perturbation theory to be polynomials in $1/a$ and $\ln(a\mu)$, with no constant term.

Minimal subtraction has the advantage that the counterterms only contain the divergences. It has the disadvantage of being purely perturbative in formulation and of depending on the choice of regulator.

4.3 Renormalization Group

The discussion in this and the next few sections will be written out for the ϕ^4 theory, with dimensional regularization as the ultra-violet regulator. It will be evident that the whole treatment can be generalized to other cases with only notational changes.

In the bare Lagrangian Eq. (14) there are exactly two parameters, m_0 and λ_0 . Renormalization is done by allowing the bare parameters to be adjusted suitably as the ultra-violet cutoff is removed, and by defining a renormalized field $\phi = \phi_0/\sqrt{Z}$, with the “wave-function renormalization” factor Z also being allowed cutoff dependence.

Suppose now that we have two renormalization schemes, which we will label as 1 and 2. The renormalized coupling and mass in each of these schemes we write as $\lambda_1, m_1, \lambda_2$ and m_2 . The wave function renormalizations are Z_1 and Z_2

Renormalized Green functions, i.e., time-ordered vacuum-expectation values of the renormalized fields, are readily related in the two schemes:

$$\begin{aligned} G_N^{(1)}(p, \lambda_1, m_1) &= Z_1^{-N/2} G_N^{(0)}(p, \lambda_0, m_0) \\ &= \left[\frac{Z_1(\lambda_1, m_1; \epsilon)}{Z_2(\lambda_2, m_2; \epsilon)} \right]^{-N/2} G_N^{(2)}(p, \lambda_2, m_2). \end{aligned} \quad (48)$$

Here $G_N^{(1)}$ and $G_N^{(2)}$ denote the renormalized N -point Green functions, while $G_N^{(0)}$ denotes the bare Green function.

Initially, we work with $\epsilon = 0$ so that all parts of this formula make sense, and then we will take the limit as $\epsilon \rightarrow 0$, so that only the renormalized Green functions can be used.

The three different Green functions in Eq. (48) are related by changes of variables:

$$(\lambda_1, m_1) \text{ to } (\lambda_0, m_0) \text{ to } (\lambda_2, m_2). \quad (49)$$

Now the renormalized Green functions are finite functions of their renormalized parameters as $\epsilon = 0$. In order for this to be consistent with the relation between the left- and the right-hand sides of Eq. (48), λ_1 and m_1 must be finite functions of λ_2 and m_2 as $\epsilon \rightarrow 0$. Furthermore Z_1/Z_2 must be a finite function of λ_2 and m_2 (and hence a finite function of λ_1 and m_1).

Thus we can write

$$G_N^{(1)}(p, \lambda_1, m_1) = z_{12}^{N/2} G_N^{(2)}(p, \lambda_2, m_2), \quad (50)$$

an equation in which all quantities are finite at $\epsilon = 0$, and $z_{12} = Z_2/Z_1$ is a calculable function of the renormalized parameters. This equation is the most fundamental expression of renormalization-group invariance. It says exactly that a change of renormalization scheme can be compensated by suitable changes in the renormalized parameters of the theory (λ and m) together with a rescaling of the renormalized field by a factor:

$$\phi_1(x) = \sqrt{z_{12}} \phi_2(x). \quad (51)$$

4.3.1 Invariance of Physical Mass and of S -matrix

The singularities of the Green functions as a function of external momenta must be at the same positions in the two schemes, by Eq. (50). In particular the physical mass of the particle is the same.

To show invariance of the S -matrix, we use the LSZ reduction formula. We need the residues of the propagator poles, which we define by:

$$G_2^{(i)} \sim \frac{ic^{(i)}}{p^2 - m_{\text{ph}}^2} \text{ as } p^2 \rightarrow m_{\text{ph}}^2. \quad (52)$$

Eq. (50) then implies that the residues are related by

$$c^{(1)} = z_{12} c^{(2)}. \quad (53)$$

The S -matrix in scheme 1 is

$$S_{2 \rightarrow N}^{(1)} = \lim_{\text{on-shell}} \frac{[c^{(1)}]^{(N+2)/2} G_{N+2}^{(1)}}{\prod \text{2 + } N \text{ propagators}}. \quad (54)$$

Each factor on the right of this equation is related to the corresponding quantity in scheme 2 by multiplication by a power of z_{12} , and the overall power of z_{12} is zero, so that the S -matrix is the same in the two schemes.

We conclude then that the S -matrix (and hence cross sections) are renormalization-group invariant.

4.4 Running Coupling and Mass

One possibility for a change of renormalization scheme is to stay within the MS scheme, but to change the value of the unit of mass μ . (Exactly similar methods can be employed in any other scheme, of course.) To keep the physics fixed, our discussion in the previous section implies that we must adjust λ and m as μ varies. That is, we must have $\lambda = \lambda(\mu)$ and $m = m(\mu)$, the running (or effective) coupling and mass.

Eq. (50) tells us the form of the relation between the Green functions with different values of μ :

$$G_N(p, \lambda(\mu'), m(\mu'), \mu') = z(\mu'/\mu, \lambda(\mu))^N G_N(p, \lambda(\mu), m(\mu), \mu), \quad (55)$$

where $z(\mu'/\mu, \lambda(\mu)) = \lim_{\epsilon \rightarrow 0} \sqrt{Z(\lambda(\mu), \epsilon)/Z(\lambda(\mu'), \epsilon)}$. By differentiating with respect to μ , we obtain the standard renormalization-group equation:

$$\mu \frac{d}{d\mu} G_N = -\frac{N}{2} \gamma(\lambda(\mu)) G_N, \quad (56)$$

where

$$\frac{1}{2} \gamma = -\mu' \frac{\partial}{\partial \mu'} z(\mu'/\mu, \lambda(\mu)) \Big|_{\mu'=\mu} \quad (57)$$

is called the “anomalous dimension” of the field ϕ . The factor $1/2$ in its definition is one standard convention. We will see the rationale for the name later. The derivative in Eq. (56) is a total derivative: it acts on the argument of the running coupling and mass as well as on the explicit μ argument of the Green function.

Given the differential renormalization-group equation Eq. (56), we can reconstruct Eq. (55)

$$G_N(p, \lambda(\mu'), m(\mu'), \mu') = G_N(p, \lambda(\mu), m(\mu), \mu) \exp \left[-\frac{N}{2} \int_\mu^{\mu'} \frac{d\hat{\mu}}{\hat{\mu}} \gamma(\lambda(\hat{\mu})) \right]. \quad (58)$$

4.5 Computation of Running Coupling and Mass

To make Eq. (58) an effective way of performing calculations, we must have a convenient way of obtaining the effective coupling $\lambda(\mu)$ and the effective mass $m(\mu)$. First we observe that the bare coupling λ_0 is renormalization-group invariant. Dimensional analysis and the knowledge that counterterms are polynomial in mass when minimal subtraction is used implies that the bare coupling has the form $\lambda_0 = \mu^\epsilon \bar{\lambda}(\lambda, \epsilon)$. Hence renormalization-group invariance of λ_0 gives

$$0 = \mu \frac{d\lambda_0}{d\mu} = \mu^\epsilon \left[\epsilon \bar{\lambda} + \mu \frac{d\lambda(\mu)}{d\mu} \frac{\partial \bar{\lambda}}{\partial \lambda} \right]. \quad (59)$$

Hence

$$\mu \frac{d\lambda(\mu)}{d\mu} = -\frac{\epsilon \lambda_0(\lambda, \epsilon)}{\partial \lambda_0 / \partial \lambda}. \quad (60)$$

The right-hand side is usually denoted by $\beta(\lambda, \epsilon)$, and it must be finite as $\epsilon \rightarrow 0$, so that the renormalization group variation of the coupling does not involve any divergences. The divergences on the right-hand side of Eq. (60) cancel, and this must occur order-by-order in perturbation theory.

When we renormalize the theory and calculate the bare coupling to some order in λ , we can obtain the renormalization-group coefficient β to the same order. Then we can use this approximation to β to compute, to some accuracy, the running coupling according to:

$$\int_{\lambda(\mu)}^{\lambda(\mu')} \frac{d\lambda}{\beta(\lambda, 0)} = \ln \frac{\mu'}{\mu}, \quad (61)$$

which is obtained by integrating the differential equation Eq. (60).

We may similarly obtain the running mass and its renormalization group coefficient from

$$\mu \frac{dm^2(\mu)}{d\mu} = \gamma_m(\lambda) m^2 \quad (62)$$

with

$$\gamma_m(\lambda) = -\beta(\lambda, \epsilon) \frac{\partial \ln m_0^2 / m^2}{\partial \lambda}. \quad (63)$$

The lack of ϵ dependence for γ_m is due to the specifics of minimal subtraction.

We may also derive a formula for the anomalous dimension:

$$\gamma(\lambda) = \beta(\lambda, \epsilon) \frac{\partial \ln Z(\lambda, \epsilon)}{\partial \lambda}. \quad (64)$$

4.6 Renormalization-Group Coefficients in ϕ^4 theory

Up to two-loop order, the bare coupling is⁴

$$\lambda_0 = \mu^\epsilon \left[\lambda + \frac{3}{\epsilon} \frac{\lambda^2}{16\pi^2} + \frac{\lambda^3}{(16\pi^2)^2} \left(\frac{9}{\epsilon^2} - \frac{17}{6\epsilon} \right) + O(\lambda^4) \right]. \quad (65)$$

It follows that the β function is

$$\begin{aligned} \beta &= \frac{-\epsilon \left[\lambda + \frac{3}{\epsilon} \frac{\lambda^2}{16\pi^2} + \frac{\lambda^3}{(16\pi^2)^2} \left(\frac{9}{\epsilon^2} - \frac{17}{6\epsilon} \right) + O(\lambda^4) \right]}{1 + \frac{6}{\epsilon} \frac{\lambda}{16\pi^2} + \frac{\lambda^2}{(16\pi^2)^2} \left(\frac{27}{\epsilon^2} - \frac{17}{2\epsilon} \right) + O(\lambda^4)} \\ &= -\epsilon \lambda + \frac{3\lambda^2}{16\pi^2} - \frac{17}{3} \frac{\lambda^3}{(16\pi^2)^2} + O(\lambda^4). \end{aligned} \quad (66)$$

4.7 Solution for Running Coupling

Given the renormalization-group equation

$$\mu \frac{d\lambda}{d\mu} = \beta(\lambda(\mu)), \quad (67)$$

and a finite-order approximation to β , we may get an approximation for the running of the coupling. We can estimate the errors in the solution, when λ is small, from knowing the order of magnitude of uncalculated higher-order terms in β .

We remove the regulator, and observe that β is proportional to λ^2 , with a positive coefficient, for small λ . For large λ , we are out of the range of accuracy of finite-order calculations. We show in Fig. 10 what happens to the running of the coupling if β has a zero at some non-zero $\lambda = \lambda^*$. For small μ , $\lambda(\mu)$ is small and approximately proportional to $1/\ln(1/\mu)$; this is a firm prediction in a region where perturbation theory is accurate. We call $\lambda = 0$ an infra-red fixed point. At large μ , $\lambda(\mu) \rightarrow \lambda^*$. We therefore call λ^* an “ultra-violet fixed point”. If β did not have a zero, then the coupling would increase without limit as $\mu \rightarrow \infty$.

4.8 Asymptotic Freedom

Suppose we have theory in which β is *negative* for small coupling. An example is QCD, where we would use the notation α_s instead of λ . Then zero coupling

⁴Terms up to five-loop order are now known.[9]

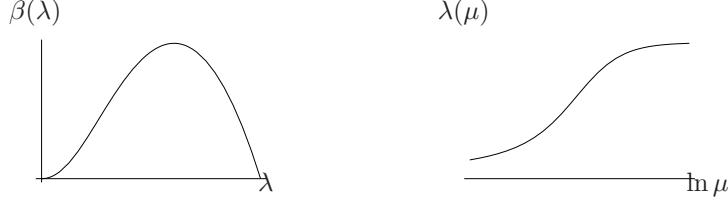


Figure 10: Running coupling in ϕ^4 on the assumption of an ultra-violet fixed point.

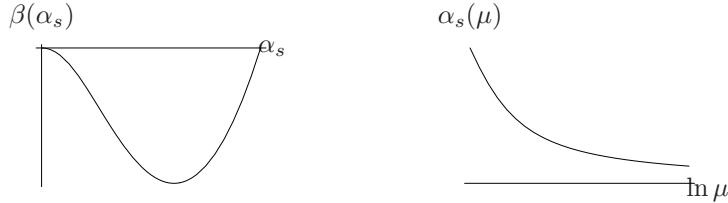


Figure 11: Running coupling in an asymptotically free theory.

is an ultra-violet fixed point; this is the defining property of an “asymptotically free” theory. As shown in Fig. 10, the coupling goes to zero as $\mu \rightarrow \infty$.

In an asymptotically free theory we may use the renormalization group to allow accurate calculations of ultra-violet dominated quantities, as we will see in a moment.

4.9 Large Momentum Behavior

Suppose we wish to calculate the large momentum behavior of a Green function, for example the two point function $G_2(p^2)$ as $|p^2| \rightarrow \infty$. We use the solution of the renormalization group equation (58) to replace a fixed μ by $\sqrt{|p^2|}$:

$$\begin{aligned}
& G_2(p^2, \lambda(\mu), m(\mu), \mu) \\
&= G_2\left(p^2, \lambda\left(\sqrt{|p^2|}\right), m\left(\sqrt{|p^2|}\right), \sqrt{|p^2|}\right) e^{\int_\mu^{\sqrt{|p^2|}} \frac{d\mu'}{\mu'} \gamma(\lambda(\mu'))} \\
&\simeq G_2\left(p^2, \lambda\left(\sqrt{|p^2|}\right), 0, \sqrt{|p^2|}\right) e^{\int_\mu^{\sqrt{|p^2|}} \frac{d\mu'}{\mu'} \gamma(\lambda(\mu'))} \\
&\simeq \frac{1}{|p^2|} G_2\left(p^2/|p^2|, \lambda\left(\sqrt{|p^2|}\right), 0, 1\right) e^{\int_\mu^{\sqrt{|p^2|}} \frac{d\mu'}{\mu'} \gamma(\lambda(\mu'))}, \tag{68}
\end{aligned}$$

where in the second line we have neglected the mass, as is sensible at large p^2/m^2 and in the last line we simply used dimensional analysis.

If we have a theory with a UV fixed point $\lambda^* \neq 0$, then the large p behavior is

$$G_2(p^2, \lambda(\mu), m(\mu), \mu) \simeq \text{constant} \times (p^2)^{-1+\frac{1}{2}\gamma(\lambda^*)}. \quad (69)$$

The power law would be obtained from the naïve considerations of dimensional analysis if the dimension of the field were increased by $\gamma/2$; this accounts for the terminology “anomalous dimension”.

If we were working in an asymptotically free theory, then $\lambda(\sqrt{|p^2|}) \rightarrow 0$ as $p^2 \rightarrow \infty$, and we would be able to use weak coupling perturbation theory to estimate the right-hand side of Eq. (68), up to an overall factor. The power law would be $1/p^2$, but modified by logarithmic corrections from the integral over the anomalous dimension function.

4.10 Coupling in asymptotically free theory

I will conclude by indicating a few properties of an asymptotically free theory, using QCD as an example. There the running coupling obeys

$$\mu \frac{d}{d\mu} \alpha_s(\mu) = -\frac{\beta_0}{2\pi} \alpha_s^2 - \frac{\beta_1}{4\pi^2} \alpha_s^3 - O(\alpha_s^4), \quad (70)$$

where in fact the first three coefficients on the right are known[10]. A solution of this equation has one constant of integration. It is common to define it by writing the solution as

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln \mu^2/\Lambda^2} - \frac{8\pi\beta_1 \ln(\ln \mu^2/\Lambda^2)}{\beta_0^3 \ln^2(\mu^2/\Lambda^2)} + O\left(\frac{\ln^2(\ln \mu)}{\ln^3 \mu}\right). \quad (71)$$

The boundary condition determining the numerical value of the constant Λ is that there is no $1/\ln^2(\mu^2/\Lambda^2)$ term. One parameter Λ determines the whole function $\alpha_s(\mu)$.

As stated earlier, one can then use the renormalization group to allow the use of perturbation theory to compute large-momentum behavior. The condition for the applicability of perturbation theory for computing behavior at some scale Q is that α_s/π is sufficiently less than unity.

4.11 Bare coupling in asymptotically free theory

One nice application of the renormalization group is to compute the true bare coupling. This means that one computes how the bare coupling behaves as the regulator is removed with α_s being held fixed. Recall that the bare coupling is expressed as a perturbation series in the renormalized coupling α_s , so fixed order perturbation theory does not directly predict the true behavior of the bare coupling. (Perturbation theory is *a priori* about the limit $\alpha_s \rightarrow 0$ with the regulator fixed.)

Let us suppose we use a lattice regulator. (We can do a similar calculation with dimensional regularization, but the result is less intuitive.) The momentum scale relevant to the bare coupling is the inverse of the lattice spacing a . We

start with the bare coupling $\alpha_{s(0)}(\alpha_s, a\mu)$ at the values of α_s and μ that we are interested in. Then we use a renormalization-group transformation to reexpress it in terms of the running coupling at $\mu = 1/a$. Asymptotic freedom then enables us to get a useful result from low order perturbation theory.

We can write the bare coupling in terms of the $\overline{\text{MS}}$ coupling in the form

$$\alpha_{s(0)} = \alpha_s + \frac{\beta_0 \alpha_s^2}{2\pi} [\ln(a\mu) + C_0] + O(\alpha_s^3), \quad (72)$$

where the constant C_0 is known. Renormalization-group invariance of the bare coupling enables us to write this as

$$\begin{aligned} \alpha_{s(0)}(\alpha_s(\mu), a\mu) &= \alpha_{s(0)}(\alpha_s(1/a), 1) \\ &= \alpha_s(1/a) + \frac{C_0 \beta_0 \alpha_s^2(1/a)}{2\pi} + O(\alpha_s^3(1/a)) \\ &= \frac{4\pi}{\beta_0 \ln 1/a^2 \Lambda^2} - \frac{8\pi \beta_1 \ln(\ln 1/a^2 \Lambda^2)}{\beta_0^3 \ln^2(1/a^2 \Lambda^2)} + \frac{8\pi C_0}{\beta_0 \ln^2(1/a^2 \Lambda^2)} \\ &\quad + \dots \end{aligned} \quad (73)$$

An important result is that the omitted terms can be omitted without affecting the $a \rightarrow 0$ limit of the renormalized theory. Hence it is sufficient to know the β function to two-loop order, and the relation between the renormalization of the lattice theory and $\overline{\text{MS}}$ renormalization at one-loop order.

For reference, note that

$$\beta_0 = 11 - \frac{2}{3}n_f; \quad \beta_1 = 51 - \frac{19}{3}n_f, \quad (74)$$

where n_f is the number of quark flavors. A reasonable fit to the measured coupling, with $n_f = 5$, is obtained with $\Lambda_{\overline{\text{MS}}} = 0.5 \text{ GeV}$. This implies that $\alpha_s(m_Z) \simeq 0.117 \pm 0.005$.

4.12 Summary

1. Renormalizable theories are renormalization-group invariant. That is, the physics they predict is unchanged under a change of the renormalization scheme, provided suitable changes in the values of the renormalized coupling and mass are made.
2. This leads to the concepts of the running coupling and mass.
3. We can use this information to compute and analyze momentum-dependence. One particular case is the computation of the true properties of the bare parameters as the regulator is removed.
4. A very important case is that of asymptotically free theories, where the running coupling goes to zero at infinite momentum.

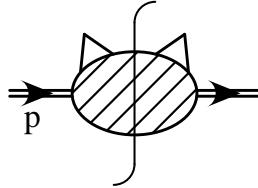


Figure 12: Hadronic part of deep-inelastic scattering.

5 Operator Product Expansion (OPE)

The theory of the strong interactions of hadrons, QCD, is asymptotically free theory. This implies that we can use the renormalization group to calculate a Green function, for example, if all its external momenta get large. However, normal physical processes never have all their momenta large, since scattering experiments have to keep their beams and targets on-shell. The operator product expansion is the simplest of a number of theorems on asymptotic behavior in field theory that enable useful predictions to be made in QCD.

A simple case is deep-inelastic scattering of an electron off a hadron: $e + p \rightarrow e' + X$. We work in the good approximation of single photon exchange, and let p^μ be the target hadron's momentum and let q^μ be the exchanged photon's momentum. We choose to sum over all possible hadronic final-states X given p^μ and q^μ . Then the strong interaction part of the cross section is obtained from the Green function depicted in Fig. 12. The deep-inelastic limit is $-q^2 \rightarrow \infty$ with $x \equiv -q^2/2p \cdot q$ fixed.

5.1 Simplest case

To see the relevant field-theoretic principles at work in their simplest form we consider the following Green function in ϕ^4 theory:

$$W(p, q) = \int d^4x d^4y_1 d^4y_2 e^{-iq \cdot x - ip \cdot y_1 + ip \cdot y_2} \langle 0 | T j(x) j(0) \phi(y_1) \phi(y_2) | 0 \rangle, \quad (75)$$

where $j = \frac{1}{2}\phi^2$ is used instead of the electromagnetic current, and the two ϕ factors are used as interpolating fields for the target. We have set the positions of one of the four fields equal to zero, to avoid having to carry around a delta-function for momentum conservation.

We will choose to take a limit in which we scale q to infinity: $q^\mu = \lambda q_0^\mu$, with $\lambda \rightarrow \infty$ at fixed q_0^μ and fixed p^μ . This is not exactly the deep-inelastic limit, but a dispersion relation can be used to relate the results of the OPE we derive in this limit to moments of deep-inelastic structure functions.⁵

⁵ The limit we take can be characterized as ‘‘Euclidean’’, in contrast to the Bjorken limit of deep-inelastic scattering, which is only possible for Minkowskian momenta. The Bjorken limit has both q^2 and $p \cdot q$ scaling as λ^2 , with $\lambda \rightarrow \infty$, whereas for Euclidean vectors one must

The OPE has the form

$$\begin{aligned}
 \text{Diagram: } &= \sum_i C_i(q) \int d^4 y_1 d^4 y_2 e^{-ip \cdot y_1 + ip \cdot y_2} \langle 0 | T \mathcal{O}_i 0 \rangle \phi(y_1) \phi(y_2) | 0 \rangle \\
 &= \sum_i C_i(q) \text{Diagram: } \overset{\bullet}{\text{O}_i} \text{,}
 \end{aligned} \tag{76}$$

where the sum runs over all local operators ($\mathcal{O}_i = \phi^2, (\partial\phi)^2, \phi^4$ etc). The predictive power of the OPE will come from the fact that the coefficient functions $C_i(q)$ have greater power law suppressions as the dimension of \mathcal{O}_i gets larger. Only a small number of terms is relevant, and the renormalization group plus finite-order perturbation theory may be used to compute the coefficients to a useful approximation. The product of two fields $\phi(y_1)\phi(y_2)$ may be replaced by a product of any number of fields of fixed momenta; these would define the target system in deep-inelastic scattering.

5.1.1 Lowest order

We can get basic intuition about how the OPE arises by examining tree graphs. For example, the lowest order graphs give:

$$\begin{aligned}
 \text{Diagram: } &+ \text{Diagram: } \\
 &= \left(\frac{i}{p^2 - m^2} \right)^2 \left[\frac{i}{(q+p)^2 - m^2} + \frac{i}{(q-p)^2 - m^2} \right] \\
 &= \left(\frac{i}{p^2 - m^2} \right)^2 \left[\frac{2i}{q^2} + \frac{2i}{(q^2)^2} \left(m^2 - p^2 + \frac{4p \cdot q^2}{q^2} \right) + O\left(\frac{1}{q^6}\right) \right] \\
 &= \frac{2i}{q^2} \text{Diagram: } \overset{\bullet}{\frac{1}{2}\phi^2} + \frac{2i}{(q^2)^2} \text{Diagram: } \overset{\bullet}{\frac{1}{2}(m^2\phi^2 - (\partial\phi)^2)} \\
 &+ \frac{8iq^\mu q^\nu}{(q^2)^3} \text{Diagram: } \overset{\bullet}{\frac{1}{2}\partial_\mu\phi\partial_\nu\phi} + \dots
 \end{aligned} \tag{77}$$

Here we have expanded in powers of the small variables, m/q , and p/q . In applications, we typically only keep the first term in the series, the term of order $1/q^2$ in Eq. (77). In the diagrams, I have used thick lines to denote the lines that are forced to carry large momentum.

have $p \cdot q$ scaling only as λ , if $q^2 \propto \lambda^2$. Asymptotic problems in Minkowski space are treated in Sterman's lectures, where they will be seen to be substantially more complex.

There is a pattern here, which we can represent in the form

(78)

On the left-hand side we have a general graph for the left-hand side of Eq. (76). On the right, we have split it into two factors: The top factor, marked by thick lines, carries large momentum, of order Q , and is called the “hard part”. The bottom factor carries momenta of low virtuality, and it is called the “soft part”. By expanding the hard part in powers of its soft external momenta, we obtain terms in the Wilson coefficients C_i . Then the soft part turns into the matrix elements of the operators \mathcal{O}_i in Eq. (76).

All the q dependence is in the coefficients $C_i(q)$, and all the p and m dependence is in the Green functions or matrix elements of the operators \mathcal{O}_i . Just as with the ultra-violet divergences, the power of q in $C_i(q)$ may be determined by dimensional analysis, as follows. Suppose $C_i(q) \propto 1/q^{N_i}$.

$$-N_i = 2 \dim(j) - 4 - \dim(\mathcal{O}_i), \quad (79)$$

where the “4” comes from the x integral for the Fourier transform. A particular consequence of this is that the leading power of q is in the term with the lowest dimension operator. This strongly restricts the number of terms

5.1.2 Is the pattern general?

We may test the pattern just conjectured by examining other graphs. Consider the case of four external low momentum lines. To lowest order we find:

+

(80)

The dots indicate terms with other operators, for example, the ones in Eq. (77), and again thick lines are those that are forced to carry large momentum.

Evidently the pattern is general, but we must remember to add in all operators, not just those with two ϕ fields.

5.1.3 Loops

The one-loop graphs for Eq. (76) are obtained from:

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} \\
 & + \text{Diagram 4} + 3 \text{ similar} + \text{Diagram 5} + O(\lambda^2). \quad (81)
 \end{aligned}$$

To economize on notation, I have exhibited the graphs with propagator corrections by replacing free by complete propagators everywhere in the tree graphs. Let us examine only the leading power ($1/q^2$), so that we only need the terms with the ϕ^2 operator. So we need to pick out the order λ terms in the OPE:

$$C(q^2) \left[\text{Diagram 6} + \text{Diagram 7} + O(\lambda^2) \right], \quad (82)$$

where we write the coefficient as the lowest order term, from Eq. (77), times a power series in λ :

$$C(q^2) = \frac{2i}{q^2} \left[1 + \lambda C^{(1)}(q^2) + O(\lambda^2) \right]. \quad (83)$$

There are some obvious correspondences. For example, graphs in Eq. (81) with an external propagator correction correspond to terms in Eq. (82) with a lowest-order coefficient times a propagator correction to the lowest-order matrix element.

However, there are some clear differences. For example, loops including the vertex for the “current” j can have ultra-violet divergences, as in the first graph on the second line of Eq. (81), while loops including the operators in the OPE can have UV divergences, as in the second graph in Eq. (82).

Moreover, in the last graph in Eq. (81), we see a non-trivial case where the loop momentum can range from being soft to hard. We cannot uniquely assign the lines to hard and soft subgraphs. This is the source of the characteristic complications of the derivation of the OPE.

But first we must understand how to define the operators; they are ultra-violet divergent and need renormalization.

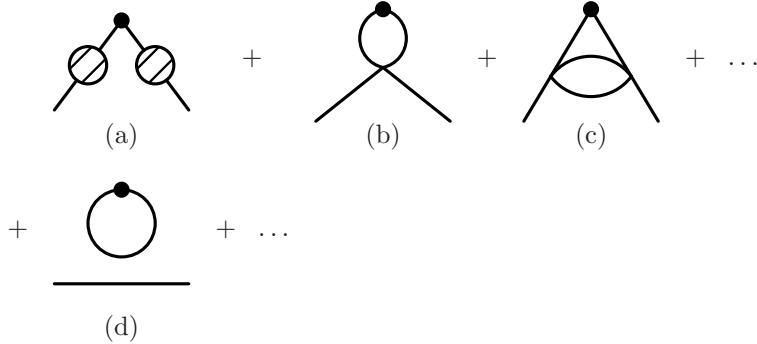


Figure 13: Graphs for the $\phi\phi$ Green function of $\frac{1}{2}\phi^2$.

5.2 Composite operators and their renormalization

In the above equations, we have seen a number of examples of composite operators, i.e., operators which are the product of fields and their derivatives at the same space-time point. There are a number of situations where we need to treat matrix elements and Green functions involving composite operators:

- To represent the coupling of QCD to electroweak fields.
- For currents and charges, as in Noether's theorem. The operators for coupling QCD to electroweak gauge bosons are Noether currents.
- In the OPE, as auxiliary constructs.

These operators, in general, can have UV divergences beyond those that are cancelled by counterterms in the QCD Lagrangian, except in the case of properly defined Noether currents for symmetries. The composite operators themselves need to be renormalized, by following the same principles as for the renormalization of the interactions.

Consider the graphs for $\langle 0|T\frac{1}{2}\phi^2 \tilde{\phi}(p_1) \tilde{\phi}(p_2)|0\rangle$, shown in Fig. 13. Some of the divergences we have already seen. For example, the one-loop subdivergence in (c) is cancelled by a interaction counterterm that we have already calculated; this gives the graph in Fig. 14(a). But some of the other divergences are new. These all include the vertex for the $\frac{1}{2}\phi^2$ operator, and they may be cancelled by extra counterterms. We now examine the interpretation of the new counterterms.

The counterterms are in fact of the form of vertices for composite operators. For example the counterterms for the overall divergences in Fig. 13(b) and (c) are a coefficient times the vertex for $\frac{1}{2}\phi^2$, as shown in Fig. 14(b). The disconnected graph Fig. 13(d) needs a counterterm proportional to the unit operator, as in Fig. 14(c).

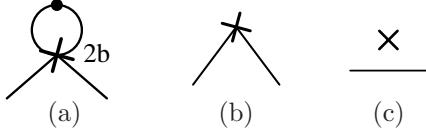


Figure 14: Counterterm graphs for Fig. 13.

It should be evident that this is a general result: We can generate renormalized Green functions of a composite operator by including counterterms of the form of coefficients times (vertices for) composite operators. It follows that we may define a renormalized $\frac{1}{2}\phi^2$ operator as

$$R\left[\frac{1}{2}\phi^2\right] = \frac{1}{2}\phi^2[1 + \delta_1] + 1\delta_2 m^2, \quad (84)$$

where the ‘1’ stands for the unit operator, and $\delta_1(\lambda, \epsilon)$ and $\delta_2(\lambda, \epsilon)$ are the counterterm coefficients. The usual power counting arguments tell us that in a renormalizable theory we need as counterterms all operators of the same and lower dimension as the operator we start with, modulo the restrictions imposed by symmetries of the theory. (In this case the relevant symmetries are Lorentz invariance and $\phi \rightarrow -\phi$.)

On the right-hand side of the OPE Eq. (76) we must use renormalized operators, so that we have finite quantities to work with. In effect, we, the users of the OPE, get to choose our definitions of the operators, and then we must prove that the coefficients can be chosen to make the OPE true.

5.3 Construction of OPE

In the first three graphs on the right-hand side of Eq. (81), we have some external propagators times a subgraph through which large momentum flows. So we may set $p = m = 0$ in the subgraph with the large momentum. This gives a q dependent factor times the vertex for $\frac{1}{2}\phi^2$ times the external propagators, i.e., we obtain purely a contribution to the $O(\lambda)$ part of the coefficient.

The only remaining graph is in the last line of Eq. (81). The loop has the value

$$-\lambda \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\epsilon)^2 [(q+k)^2 - m^2 + i\epsilon]}. \quad (85)$$

First observe that in a region where k is fixed as q gets large, we have a contribution

$$\frac{2i}{q^2} \frac{i\lambda}{2} \int_{\text{small}} \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\epsilon)^2}. \quad (86)$$

We have organized the normalization factors so that this is manifestly the lowest-order coefficient times the one-loop Green function of the $\frac{1}{2}\phi^2$ operator, the second graph in Eq. (82). This is a completely expected contribution.

The one-loop part of operator Green function also has a contribution from large k , where the above argument does not work. The original graph has a contribution when k is of order q or bigger. To avoid double counting we take that graph and subtract the one-loop Green function of $\frac{1}{2}\phi^2$ times the lowest-order Wilson coefficient. It is now legitimate to neglect p and m , so that we have

$$-\lambda \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{1}{(k^2)^2 (q+k)^2} - \left[\frac{1}{(k^2)^2} - \text{UV counterterm} \right] \frac{1}{q^2} \right\}. \quad (87)$$

This represents the original graph minus a corresponding term that we already know is on the right of the OPE. The effect is that we have subtracted the low momentum behavior, and the infra-red divergence that would otherwise be present when $m = p = 0$ has cancelled. The result it now an $O(\lambda)$ contribution to the coefficient.

The generalization of the above results is that the $q \rightarrow \infty$ of the left hand-side of Eq. (76) is a sum of terms with a representation in Eq. (78). The $1/q^2$ coefficient of $\frac{1}{2}\phi^2$ is a subtracted sum of graphs for $\langle 0 | T j(q) j \tilde{\phi}(p_1) \tilde{\phi}(p_2) | 0 \rangle$, 1PI in the $\tilde{\phi}$ legs, with m and the ps set to zero, and with subtractions made to cancel the resulting IR divergences.

5.4 Proof

It is not too hard to sketch a proof. For each graph Γ for the left-hand-side of the OPE, we define a remainder

$$r(\Gamma) = \Gamma - \text{UV counterterms} - \text{Subtractions for leading } q \rightarrow \infty \text{ behavior.} \quad (88)$$

The subtractions for the large q behavior can be constructed in much the same fashion as renormalization counterterms. Much of the technology is very similar, in fact. If we just subtract the leading power ($1/q^2$ times logarithms of q), then the remainder is of order $1/q^3$ times logarithms. Subtracting more terms gives a correspondingly smaller remainder. For the 1-loop graph we just considered, we write

$$\begin{aligned} r(\text{1 loop graph}) &= \text{1 loop graph} + C(\text{tree}) R[\text{1 loop ME}] \\ &\quad + C(\text{1 loop graph}), \end{aligned} \quad (89)$$

where now $C(\Gamma)$ means that we take the negative of the subgraph Γ with $m = p = 0$, *after subtraction of IR divergences*.

There are of course the usual difficult issues of showing that the subtractions actually cancel the divergences, and that the standard power-counting criteria are correct; in particular that the remainder is actually suppressed by the claimed power of q compared with the original graphs.

Then we sum over all graphs and show that the subtraction terms have the form of coefficients times Green functions of operators.

5.5 Use of renormalization group

Consider the leading term in the OPE Eq. (76) for connected graphs. Let us write it as

$$C(q^2, \lambda(\mu), \mu) G_{\frac{1}{2}\phi^2}(p, \lambda(\mu), \mu, m). \quad (90)$$

When $q \rightarrow \infty$ with the other variables fixed, we get logarithms of q/μ in higher order contributions to the coefficient C . These ruin the accuracy of fixed-order calculations.

But we may use the renormalization group to set $\mu = |q|$. We can then use a low order calculation of $C(q^2, \lambda(|q|), |q|)$, provided only that $\lambda(|q|)/16\pi^2$ is small. A knowledge of the anomalous dimension of the fields and of $\frac{1}{2}\phi^2$ enables us to express $G_{\frac{1}{2}\phi^2}$ at $\mu = |q|$ in terms of its value at fixed μ .

The important facts are that we have separated the original Green function, which depends on two widely different scales q and m (or p), into two factors, each of which depends on only one of the scales. We can now change the renormalization mass μ independently in each factor, and a change of μ suffices to remove the logarithms. It is essential for this that the Wilson coefficient C have no IR divergences, after subtraction.

If we left μ fixed we would have a series of the form

$$\begin{aligned} C &= \frac{\text{constant}}{q^2} \left[1 + \lambda \left(\# \ln \frac{q^2}{\mu^2} + \# \right) \right. \\ &\quad \left. \lambda^2 \left(\# \ln^2 \frac{q^2}{\mu^2} + \# \ln \frac{q^2}{\mu^2} + \# \right) + \dots \right], \end{aligned} \quad (91)$$

where, the ‘#’s stand for numerical constants.

5.6 Renormalization group equation

By expressing the renormalized operators in terms of bare operators, we can find RG equations

$$\mu \frac{d\phi}{d\mu} = \mu \frac{dZ^{-1/2}}{d\mu} \phi_0 = -\frac{1}{2}\gamma(\lambda(\mu)) \phi, \quad (92)$$

$$\mu \frac{dR[\phi^2]}{d\mu} = \mu \frac{dZ_{\phi^2}}{d\mu} \phi_0^2 = \gamma_{\phi^2}(\lambda(\mu)) R[\phi^2], \quad (93)$$

with

$$\gamma_{\phi^2}(\lambda(\mu)) = \beta(\lambda, \epsilon) \frac{\partial \ln Z_{\phi^2}(\lambda, \epsilon)}{\partial \lambda}. \quad (94)$$

Here, Z_{ϕ^2} is the renormalization factor of the ϕ^2 operator: $Z_{\phi^2} = (1 + \delta_1)Z^{-1}$ in the notation of Eq. (84). We have ignored, for simplicity, the term with the unit operator; it is not needed if we treat connected Green functions only.

Similar equations can be derived for the other composite operators.

From these equations follows a renormalization group equation for the Wilson coefficient:

$$\mu \frac{d}{d\mu} C(q^2, \lambda(\mu), \mu) = [2\gamma_j(\lambda) - \gamma_{\phi^2}(\lambda)] C, \quad (95)$$

where γ_j is the anomalous dimension of the “current” j , which in our simplified example happens also to be the operator ϕ^2 .

The final form of the leading term for the large q behavior of the left-hand side of Eq. (76) is

$$e^{\int_\mu^{|q|} \frac{d\mu'}{\mu'} [-2\gamma_j(\lambda(\mu')) + \gamma_{\phi^2}(\lambda(\mu'))]} C(q^2, \lambda(|q|), |q|) G_{\frac{1}{2}\phi^2}(p, \lambda(\mu), \mu, m). \quad (96)$$

Consequences of this result and its generalizations include

- In a theory with a non-zero UV fixed point, we obtain a power law not equal to the naïve one, but we cannot do perturbative calculations.
- In an asymptotically free theory, like QCD, we obtain the power law given by dimensional analysis, but with logarithmic corrections. The errors are under control, since we can expand the anomalous dimensions and the Wilson coefficients in powers of a small parameter α_s . The operator matrix elements cannot be calculated perturbatively; at the present state of the art, they must be measured experimentally. However, the matrix elements are universal: the same ones can be used in treating many different processes.
- If j is a conserved current (e.g., an electromagnetic current), then its anomalous dimension is exactly zero, $\gamma_j = 0$.

6 Renormalization etc in the standard model

In this section, I will summarize the issues that arise in generalizing the results of the previous sections to the standard model, and indeed to a general quantum field theory. Among these issues are

- There are a lot of extra couplings. Is the theory nevertheless renormalizable? What are the RG equations?
- There are many symmetries, some of them broken. How do we treat renormalization, the RG and the OPE in the presence of: global symmetries, local symmetries, spontaneous symmetry breaking? What are the special problems that arise with chiral symmetries?
- In particular there is the possibility of “anomalous breaking” of symmetries.
- A particular issue of importance in QCD is to determine which operators we need to use in the OPE. It needs a highly non-trivial proof to show that we can omit all but gauge-invariant operators.

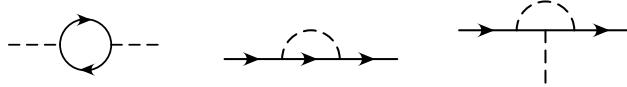


Figure 15: Divergent one-loop graphs in Yukawa theory that have counterterms formed by renormalizing the original Lagrangian.

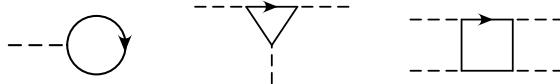


Figure 16: Divergent one-loop graphs in Yukawa theory that do not have counterterms corresponding to terms in the original Lagrangian.

I will only have space to present a summary of the issues.

The Lagrangian for the standard model, with its $SU(3) \times SU(2) \times U(1)$ symmetry, can be written in a very compact form:

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}^{\alpha\beta} + \bar{\psi}(iD^\mu - M)\psi + D^\mu\phi^\dagger D^\mu\phi - V(\phi, \phi^\dagger) + \bar{\psi}\Gamma\psi\phi + h.c. \quad (97)$$

Here we use a standard notation, for the gauge, fermion and Higgs fields. All these fields have been put together into big vectors, so that there are implicitly a lot of summed indices in each term.

I will address in turn the different issues involved in renormalizing this theory by examining a series of simplified model theories. The issues are generic to quantum field theories, and have nothing to do with the specific form of the standard model.

6.1 Need for extra interactions

Consider a Yukawa model of a Dirac field and a real scalar field:

$$\mathcal{L}_{v1} = \frac{1}{2}\partial\phi^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\partial^\mu - M)\psi + g\bar{\psi}\psi\phi. \quad (98)$$

We start off by finding the divergent one-loop graphs.

The scalar self energy, the fermion self energy and the fermion-scalar vertex (Fig. 15) have degrees of divergence 2, 1, 0, and thus their divergences can be cancelled by counterterms proportional to terms in Eq. (98), just as in the ϕ^4 theory. But in addition there are other divergent graphs, shown in Fig. 16. These need counterterms proportional to ϕ , ϕ^3 , and ϕ^4 , terms not in Eq. (98). Note that the first two graphs have degrees of divergence above zero, but there are no Lorentz-invariant operators of an appropriate dimension except the ones listed.

We cancel all these divergences by renormalizing the original Lagrangian and then adding some extra terms:

$$\begin{aligned} \mathcal{L}_{v2} = & \frac{1}{2}Z_\phi\partial\phi^2 - \frac{1}{2}m_B^2\phi^2 + Z_\psi\bar{\psi}i\partial^\mu\psi - M_B\bar{\psi}\psi + g_B\bar{\psi}\psi\phi \\ & - \lambda_{B1}\phi - \lambda_{B3}\phi^3 - \lambda_{B4}\phi^4. \end{aligned} \quad (99)$$

Since we have added 3 new couplings, we must ask “Does this process ever stop? Do we have to keep adding even more couplings as we go to higher order?”

The answer is that no extra couplings are needed. We notice first that at one loop order, graphs with more external lines than those we listed are convergent. Then we observe that not only the original couplings but also the extra ones have a mass dimension that is zero or larger. Let us apply dimensional analysis. The degree of UV divergence of a graph equals the dimension of its integral, so that

$$\frac{\text{dim. of graph}}{D(\Gamma)} = \frac{\text{dim. of couplings}}{\delta(\Gamma)} + \frac{\text{deg. of divergence}}{\Delta(\Gamma)} \quad (100)$$

The degree of divergence is therefore $D(\Gamma) - \delta(\Gamma)$. The counterterm is a polynomial in momenta of this degree. Now, the couplings in the counterterms are the coefficients of the terms in the polynomial. Since the whole counterterm polynomial is to be added to the graph, its dimension is the same as that of the graph, i.e., $D(\Gamma)$. Hence the dimensions of the couplings of the counter-terms range from $D(\Gamma)$, for a term that is independent of momentum, down to $D(\Gamma) - \Delta(\Gamma) = \delta(\Gamma)$, for a term with the maximum power of momentum. Hence the dimensions of the counterterm couplings are all at least as large as $\delta(\Gamma)$, which is the total of the dimensions of the couplings in the original graph.

Hence if we start with couplings of non-negative dimension, all the counterterm couplings have non-negative dimension. There are a finite number of such terms, since the fields have positive dimension, and thus the process of generating new counterterm vertices terminates. Indeed, in the second version of the Yukawa Lagrangian, Eq. (99), we have all possible terms with couplings of non-negative dimension, given that they must be Lorentz invariant and parity-invariant. The last requirement prohibits terms with γ_5 s in them, and is obeyed because the original Lagrangian is parity-invariant.

One important result follows, that if we write down a candidate Lagrangian for a theory of physics, then we may be forced to extend it by the addition of extra terms in order to get a renormalizable theory.

6.2 General results on renormalizability

We have the following cases

- All couplings in the initial candidate Lagrangian have dimension ≥ 0 . Then either

No new couplings are needed, as in the ϕ^4 theory.

or

We need new couplings, all of dimension ≥ 0 . But the number of such terms is finite.

Such theories are called “renormalizable” or “renormalizable with the addition of extra couplings”.

- At least one coupling has dimension < 0 . Then for each Green function, we can get a divergence of arbitrarily high degree by going to a high enough order of perturbation theory. We need an infinite collection of counterterms, and such theories are termed “non-renormalizable”. The classic physical case is the 4-fermion theory of weak interactions.

A special case of a renormalizable theory is a super-renormalizable theory, where only a finite number of 1PI *graphs* need counterterms for overall divergences. This happens when all of the interactions have strictly positive dimension, as in ϕ^4 theory in less than four space-time dimensions, but it does not happen in any interacting four dimensional theory.⁶

It is also possible to get zeros in the coefficients of divergences, so that fewer counterterms are needed than indicate by the power-counting diagnosis. This is a property of specific theories, particularly supersymmetric theories.

It is also possible that the situation may change beyond perturbation theory. Even so, very powerful results are needed to convert an theory that is apparently non-renormalizable theory by the power-counting criterion to a genuinely renormalizable theory. Our experience with non-renormalizable theories of physics is restricted to weak interactions and to gravity. In the first case the non-renormalizable theory is an approximation to the true theory, but only in a certain domain. In the second case, we still have no complete, accepted and useful theory of quantum gravity.

6.3 Renormalization group and OPE with many couplings

The principles of the renormalization group that I explained in Sect. 4 can be applied to any field theory. The main difference is that we obtain a β function for each coupling, and that it is generally a function of all the couplings. Thus the equation for the running of the coupling becomes an array of equations. For example in the Yukawa theory Eq. (99), we have

$$\mu \frac{d}{d\mu} \begin{pmatrix} \lambda_4 \\ g \end{pmatrix} = \mu \frac{d}{d\mu} \begin{pmatrix} \beta_4(\lambda_4, g) \\ \beta_g(\lambda_4, g) \end{pmatrix}. \quad (101)$$

(The dimensional coupling λ_3 does not enter into the RGEs for the dimensionless couplings.) Similar equations can be written for the dimensional couplings and the masses, and for the anomalous dimensions.

Anomalous dimensions for the fields become functions of the several couplings, as do anomalous dimensions for composite operators.

Furthermore, when one obtains the OPE in a general theory one must consider all operators of the appropriate dimension, and the renormalization-group equation will mix all the operators of a given dimension. But once that is realized, all the same principles apply as in the ϕ^4 theory.

⁶ An apparent counterexample is ϕ^3 theory. However, its energy density is not bounded below, and the theory is unstable against decay of the vacuum; it is therefore unphysical. However, the theory is a useful source of exercises on perturbation theory, where the difficulties are not manifest.

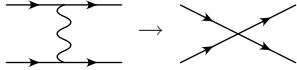


Figure 17: A 4-fermion interaction is a low energy approximation to W -exchange.

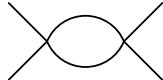


Figure 18: Quadratically divergent graph in 4-fermion theory.

6.4 The physical significance of non-renormalizable theories

The above discussion appears to indicate that non-renormalizable theories are inadmissible as theories of physics, on the grounds that one must introduce an infinite collection of counterterms to renormalize them. The inherent arbitrariness in choosing the finite parts of counterterms then implies that an infinite set of parameters is needed to specify the theory.

However, this assumes that a field theory must be used with the UV cut-off removed. However, all that is necessary is that the momentum scale of the cut-off is sufficiently much bigger than the scales currently accessible to experiments. More generally, one may suppose that there is some true fundamental theory of physics, and all that one can directly observe are low energy approximations to this true theory.

The classic example are the semi-leptonic weak interactions, as illustrated in Fig. 17. At low energies, the W 's propagator can be approximated by a constant: $1/(q^2 - m_W^2) \rightarrow -1/m_W^2$. Thus one obtains an effective 4-fermion coupling $G_F \propto g^2/m_W^2$. The negative dimension of G_F is exactly the signature of a non-renormalizable theory. Since an exchange of a light particle, like the photon, will give a much larger propagator, weak interaction amplitudes are suppressed by a power q^2/m_W^2 .

From this point of view, a characteristic of non-renormalizable interactions is that in the region where they are a good approximation to reality they are weak. Perturbation theory in the non-renormalizable coupling can be considered as an expansion in powers of the typical momentum scale of a quantity being calculated divided by a large scale like M_W . The large scale is effectively a cut off on the non-renormalizable theory.

The divergences in higher order graphs for a non-renormalizable theory, as in Fig. 18, come from regions of large momentum. That is precisely where the 4-fermion vertices are bad approximations to the true theory. We can choose to add counterterms to renormalize the graph, and the finite parts of the counterterms will be determined by the true theory. After the counterterms are added,

perturbation theory for the non-renormalizable theory is an expansion in powers of q^2/m_W^2 . The arbitrariness in the finite parts affects higher order terms in this expansion, so we are not directly bothered by the infinite number of the counterterms.

If a non-renormalizable theory is a theory of physics, then it indicates a momentum scale at which it breaks down, which is where its interactions stop being small corrections to renormalizable interactions. For the 4-fermion theory, this is of the order of 100 GeV, which is exactly where the Weinberg-Salam theory must be used.

The other established non-renormalizable theory of physics is general relativity, the theory of gravity. The scale at which it must breakdown is of the order of Planck mass. This is far above all experimentally accessible scales, and the quantum effects of general relativity are negligible in normal situations, for example in atomic and nuclear physics. Gravity is intrinsically many orders of magnitude weaker than even the weak interactions at energy scales of a GeV.

Of course, gravity dominates the physics of large scale systems, like the earth, the solar system and the universe. But in these cases we are dealing only with the classical-field limit of gravity; quantum gravity is irrelevant. For the long range forces in large systems, strong and weak interactions are unimportant because they have a finite range, and electromagnetic interactions are unimportant because non-zero electric charges are screened in large systems.

A renormalizable theory, in contrast to a non-renormalizable theory, does not contain indications of its own breakdown; it is a self-consistent theory when the UV cut-off is removed. (To be fair there are indications that non-asymptotically free theories may not exist: For example, the bare coupling in QED becomes strong if the UV cut-off is at about the Planck scale; this, at the least, implies that perturbation theory is inadequate to treat the continuum limit, unlike the case of an asymptotically free theory.)

6.5 Symmetries

A theory like the standard model has many symmetries, and it is necessary to prove, as far as possible, that the only counterterms that are necessary to renormalize the theory preserve the symmetry.

A simple case is a global symmetry, such as the symmetry $\phi \rightarrow \phi e^{i\omega}$ of the following Lagrangian of a complex scalar field:

$$\partial\phi^\dagger\partial\phi - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \quad (102)$$

By Noether's theorem this symmetry implies the existence of a conserved abelian charge, together with an associated conserved current. It is not hard to show that all the counterterms needed are symmetric. For example, no mass counterterm proportional to ϕ^2 is needed (as opposed to $\phi^\dagger\phi$). The simplest proof is to observe that the free propagator for a complex field is directed: it has an arrow on it. The interaction has two lines entering and two lines leaving it, and so Green functions are zero if they have unequal numbers of lines entering and leaving them, like Fig. 19.

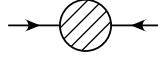


Figure 19: These graphs are zero in the theory specified by Eq. (102).

It is also possible to show that the same result on the counterterms is true if there is spontaneous breaking of the symmetry, as would occur if m^2 were negative. The simplest proofs of these theorems are made by observing that the symmetries are still valid when a UV regulator is applied to the theory. Corresponding result hold to restrict the operators that appear in the OPE.

But chiral symmetries are not necessarily preserved. There is a theorem that normal UV regulators break them. For example, one cannot define a Dirac γ_5 matrix for a dimensionally regulated theory without either losing some properties of the 4-dimensional γ_5 that are needed to prove chiral invariance or losing consistency. So chiral symmetries can only be preserved, if at all, in the limit that the regulator is removed. In general, one expects “anomalies”. There are known criteria for the cancellation of such anomalies, and one constraint of the fermion content of the standard model is that anomalies cancel; the weak and electromagnetic part of the gauge group is chiral, since the weak bosons couple differently to left- and right-handed fermions.

6.6 Gauge theories

Consider QED, whose Lagrangian, with a gauge -fixing term, is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(iD^\mu - M)\psi - \frac{1}{2\xi}\partial \cdot A^2, \quad (103)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi$. Apart from the gauge-fixing term, which we have chosen to be $-\partial \cdot A^2/2\xi$, the Lagrangian is invariant under the local transformation

$$\begin{aligned} A_\mu(x) &\rightarrow A_\mu(x) + \partial_\mu\omega(x), \\ \psi(x) &\rightarrow e^{-ie\omega(x)}\psi(x). \end{aligned} \quad (104)$$

Now, the power counting criterion allows counterterms proportional to A_μ^2 and to $(A_\mu^2)^2$, as well as to those in the original Lagrangian; i.e., it allows a photon mass term and a photon self-interaction. These terms are not gauge invariant, and in fact they are not generated as counterterms. But since the Lagrangian after gauge fixing is not gauge-invariant, a simple appeal to gauge invariance is not sufficient to provide a proof.

Even so, it can be shown that QED can be renormalized by the following transformation of Eq. (103):

$$\mathcal{L} = -Z_3\frac{1}{4}F_{\mu\nu}^2 + Z_2\bar{\psi}(iD^\mu - M_0)\psi - \frac{1}{2\xi}\partial \cdot A^2. \quad (105)$$

No renormalization of the gauge fixing term is needed, and the covariant derivative is unchanged, if expressed in terms of the renormalized fields and couplings. It can be expressed in terms of the bare fields: $D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi = \partial_\mu\psi + ie_0A_\mu^{(0)}\psi$, where the bare coupling is $e_0 = Z_3^{-1/2}e$.

After renormalization, the Lagrangian with the gauge-fixing term omitted is still gauge-invariant.

A number of other results are needed in order to demonstrate that the theory is a valid theory. The basic problem is that there are unphysical degrees of freedom and it must be proved that they decouple from all of the genuine physics. The proofs rely on gauge invariance, so that the proof that renormalization preserves gauge invariance is necessary if we are to prove that the unphysical degrees of freedom continue to decouple after renormalizing the theory.

First we define the concept of a *physical state*. For scattering states this is one where the photon polarizations obey $k \cdot \epsilon = 0$. For a single photon, this still allows 3 polarizations per momentum state, so in addition we identify states that differ by a gauge transformation. That is, we consider equivalent states which differ by changing the polarization of a photon state by $\epsilon_\mu \rightarrow \epsilon_\mu + ak_\mu$. (The number a is arbitrary and may differ from photon to photon.)

It is necessary to show that no contribution to physical matrix elements is made by photons with scalar polarization ($\epsilon_\mu \propto k_\mu$). It is also necessary to prove unitarity. That is, if \mathcal{A} and \mathcal{B} are any two physical operators and $|\text{phys1}\rangle$ and $|\text{phys2}\rangle$ are any physical states, then

$$\langle \text{phys 1} | \mathcal{A} \mathcal{B} | \text{phys 2} \rangle = \sum_{n \text{ physical}} \langle \text{phys 1} | \mathcal{A} | n \rangle \langle n | \mathcal{B} | \text{phys 2} \rangle. \quad (106)$$

Finally, it must be shown that physical matrix elements are independent of the method of gauge-fixing, which implies that they are independent of the arbitrary gauge-fixing parameter ξ . A physical matrix element is a matrix element of a gauge invariant operator between physical states. One kind of physical matrix element is an S -matrix element between physical states.

To show these results, we need the Ward identities. Interestingly, these results continue to be true if a photon mass is added to the Lagrangian, even though that breaks gauge invariance.

It can also be shown that the OPE on physical states needs only gauge-invariant operators:

$$\langle \text{phys 1} | \tilde{j}(q)j(0) | \text{phys 2} \rangle = \sum_i C_i(q) \langle \text{phys 1} | \mathcal{O}_i | \text{phys 2} \rangle, \quad (107)$$

where the sum is restricted to gauge-invariant operators \mathcal{O}_i . It can be shown that other operators do appear in the OPE, but that when we restrict to physical states they give zero.

6.7 Non-abelian gauge theories

Similar results can be proved in non-abelian gauge theories, including QCD by itself, as well as the standard model. The proofs get much harder, partly because

of the presence of Faddeev-Popov ghost fields in the gauge-fixed Lagrangian. Although the Lagrangian is not gauge-invariant, it is invariant under a kind of global supersymmetry, BRST symmetry. Since BRST transformations are non-linear, the methods that apply to ordinary global symmetries do not work without modification.

One complication is that the gauge transformations are renormalized, unlike the case of QED. The covariant derivative is written $D_\mu\psi = \partial_\mu\psi + ig_0A_\mu^{(0)\alpha}t_\alpha\psi$, where t_α are the generating matrices for the gauge group acting on the matter field. But no longer do we have g_0 equal to $Z_3^{-1/2}g$.

6.8 Summary

- The key property that makes a theory renormalizable is the that is has couplings of non-negative dimension: $\dim(\text{couplings}) \geq 0$.
- Ordinary global symmetries of the Lagrangian are preserved under renormalization, even when there is spontaneous breaking of the symmetry by the vacuum.
- But this is not true for chiral symmetries unless anomaly cancellation conditions are satisfied.
- Gauge-invariance is preserved under renormalization, but the proof is highly non-trivial. Gauge-invariance is vital to the proof of unitarity and of other properties of physical states.

Acknowledgments

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A Useful integrals etc.

The Gamma function is defined by

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t}. \quad (108)$$

It satisfies:

$$\Gamma(z+1) = z\Gamma(z). \quad (109)$$

When $z = n > 0$, an integer, $\Gamma(n) = (n-1)!$. $\Gamma(z)$ has poles at all negative and zero values of z . The Taylor series expansion about $z = 0$ is

$$\Gamma(z) = \frac{1}{z} [1 - \gamma_E z + O(z^2)], \quad (110)$$

where $\gamma_E = 0.5772\dots$ is Euler's constant. Also

$$\Gamma(1/2) = \pi^{1/2}. \quad (111)$$

The Beta function is defined by

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1}. \quad (112)$$

Hence:

$$\begin{aligned} \int_0^\infty dx x^\alpha (x+A)^\beta &= A^{1+\alpha+\beta} B(\alpha+1, -\beta-\alpha-1) \\ &= A^{1+\alpha+\beta} \frac{\Gamma(\alpha+1)\Gamma(-\beta-\alpha-1)}{\Gamma(-\beta)}. \end{aligned} \quad (113)$$

Feynman parameters:

$$\prod_{j=1}^N \frac{1}{A_j^{\alpha_j}} = \prod_{j=1}^N \left(\int_0^1 dx_j x_j^{\alpha_j-1} \right) \frac{\Gamma\left(\sum_j \alpha_j\right)}{\prod_j \Gamma(\alpha_j)} \frac{\delta\left(1 - \sum_{j=1}^N x_j\right)}{\left(\sum_{j=1}^N A_j x_j\right)^{\sum \alpha_j}}, \quad (114)$$

Spherically symmetric n -dimensional Euclidean integrals:

$$\int d^n k f(|k|) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty dk k^{n-1} f(k). \quad (115)$$

Minkowski space integrals: All with the metric where $a \cdot b = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$.

$$\int d^n k \frac{1}{(-k^2 - 2p \cdot k + C)^\alpha} = \frac{i\pi^{n/2} \Gamma(\alpha - n/2)}{\Gamma(\alpha)} (C + p^2)^{n/2-\alpha}, \quad (116)$$

$$\int d^n k \frac{k^\mu}{(-k^2 - 2p \cdot k + C)^\alpha} = \frac{i\pi^{n/2} \Gamma(\alpha - n/2)}{\Gamma(\alpha)} (C + p^2)^{n/2-\alpha} (-p^\mu), \quad (117)$$

$$\begin{aligned} \int d^n k \frac{k^\mu k^\nu}{(-k^2 - 2p \cdot k + C)^\alpha} &= \frac{i\pi^{n/2}}{\Gamma(\alpha)} (C + p^2)^{n/2-\alpha} \\ &\left[\Gamma(\alpha - n/2) p^\mu p^\nu - \frac{1}{2} \Gamma(\alpha - 1 - n/2) g^{\mu\nu} (C + p^2) \right], \end{aligned} \quad (118)$$

$$\begin{aligned} \int d^n k \frac{k^2}{(-k^2 - 2p \cdot k + C)^\alpha} &= \frac{i\pi^{n/2}}{\Gamma(\alpha)} (C - p^2)^{n/2-\alpha} \\ &\left[\Gamma(\alpha - n/2) p^2 - \frac{n}{2} \Gamma(\alpha - 1 - n/2) (C + p^2) \right]. \end{aligned} \quad (119)$$

B Problems

1. In this and the next four problems, you will calculate the one-loop counterterms for the $(\phi^3)_6$ theory, that is ϕ^3 theory in space-time dimension 6. When dimensional regularization is used, so that the dimension of space time is $n = 6 - \epsilon$, the Lagrangian is

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\mu^{\epsilon/2}g}{3!}\phi^3 \\ & + \frac{1}{2}\Delta Z(\partial\phi)^2 - \frac{1}{2}\Delta m^2\phi^2 - \frac{\mu^{\epsilon/2}\Delta g}{3!}\phi^3 - \Delta h\phi.\end{aligned}\quad (120)$$

Show that the counterterms indicated are all the ones that are needed. Note the presence of an additional term $-\Delta h\phi$ linear in the field.

2. Compute the values of all the one-loop counterterms. Assume that the coupling renormalization Δg , the wave function renormalization ΔZ , and the mass renormalization Δm^2 are defined by minimal subtraction. *But define the one-point counterterm Δh to cancel the tadpole graphs exactly, so that $\langle 0|\phi|0 \rangle = 0$.*

3. What is the renormalization-group equation for this theory? You will need to show, by consideration of the vacuum expectation value of the field, $\langle 0|\phi|0 \rangle$, and its renormalization condition, that no term involving a coupling $h\phi$ is needed.

4. Find the renormalization-group coefficients β , γ , and γ_m for this theory.

5. Still in ϕ^3 theory in six space-time dimensions, compute the physical mass and the residue of the particle pole in the propagator, both to order g^2 . Your results should be expressed in terms of the renormalized mass and coupling in the MS scheme.

6. The dependence of a Green function on the unit of mass μ in each order of perturbation theory is a polynomial in $\ln \mu$. Thus one can write the perturbation series as:

$$\sum_N \lambda_R^N \sum_{n=0}^{n_{\max}(N)} a_{Nn} (\ln \mu)^n. \quad (121)$$

(This polynomial dependence will follow from the methods you are about to use.) The degree of the polynomial is $n_{\max}(N)$, and the subject of this and the next problem is to determine some of its properties. (Although the ideas are the same in any theory, assume that we are working in $(\phi^4)_4$ theory.)

(a) Show from the renormalization group equation that $n_{\max}(N+1) = 1 + n_{\max}(N)$.

(b) Hence show that $n_{\max}(N)$ = number of loops, with the number of loops being $N - 1$ for the connected 4-point function.

7. (a) With the same set up as in the previous problem, define the leading coefficient in N th order to be $a_{Nn_{\max}}$. Find a recurrence relation between the leading coefficient at order $N+1$ and at order N . It will involve the one-loop coefficient in the β function.

(b) Hence (still in $(\phi^4)_4$ theory) find the value of $a_{N,N-1}$ for the (connected) four-point Green function.

8. Suppose that the bare coupling in ϕ^4 theory in two different renormalization prescriptions is given by

$$g_0 = \mu^\epsilon \left[g_1 + g_1^2 \left(\frac{A_{11}}{\epsilon} + A_{10}(\epsilon) \right) + g_1^3 \left(\frac{A_{22}}{\epsilon^2} + \frac{A_{21}}{\epsilon} + A_{20}(\epsilon) \right) + \dots \right], \quad (122)$$

$$g_0 = \mu'^\epsilon \left[g_2 + g_2^2 \left(\frac{B_{11}}{\epsilon} \right) + g_2^3 \left(\frac{B_{22}}{\epsilon^2} + \frac{B_{21}}{\epsilon} \right) + \dots \right]. \quad (123)$$

(Scheme 2 would be minimal subtraction.) The coefficients A_{10} and A_{20} are functions of ϵ , analytic at $\epsilon = 0$, and all the other coefficients are constants.

(a) Express the renormalized coupling, g_1 , of scheme 1 as a power series in the renormalized coupling, g_2 , of scheme 2. You will be able to obtain the series up to the g_2^3 term from the information given.

(b) What conditions must the coefficients satisfy in order that the renormalized couplings both be finite.

(c) Show that in particular the 2-loop double pole term (A_{22}/ϵ^2 or B_{22}/ϵ^2) is determined completely by the 1-loop single pole.

9. Consider ϕ^4 theory (in 4 space-time dimensions). Let λ be the coupling when the minimal subtraction scheme (MS) is used. Let the coupling in another scheme be related to it: $\lambda_1 = \lambda_1(\lambda)$.

(a) What is the renormalization-group β function in this other scheme? Call this function $\beta_1(\lambda_1)$. (The answer should involve the function $\lambda_1(\lambda)$ and β in the MS scheme.)

(b) Assume that $\lambda_1(\lambda)$ can be expanded in a power series in λ , with first term λ . Show that the first two terms in $\beta_1(\lambda_1)$ have the same coefficients as in $\beta(\lambda)$.

10. Hard problem!

(a) Compute all the two-loop renormalization coefficients in ϕ^4 theory.

(b) Hence show that

$$\begin{aligned}\lambda_0 &= \mu^\epsilon \left[\lambda + \frac{3}{\epsilon} \frac{\lambda^2}{16\pi^2} + \frac{\lambda^3}{(16\pi^2)^2} \left(\frac{9}{\epsilon^2} - \frac{17}{6\epsilon} \right) + O(\lambda^4) \right], \\ m_0^2 &= m^2 \left[1 + \frac{1}{\epsilon} \frac{\lambda}{16\pi^2} + \frac{\lambda^2}{(16\pi^2)^2} \left(\frac{2}{\epsilon^2} - \frac{5}{12\epsilon} \right) + O(\lambda^3) \right], \\ Z &= 1 - \frac{1}{12\epsilon} \frac{\lambda^2}{(16\pi^2)^2} + O(\lambda^3).\end{aligned}$$

Hints:

- i If you do the integrals in the most obvious way they will be quite complicated. But you can take advantage of the fact that, for example, the counterterms for the coupling are independent of the mass, and so set $m = 0$ in the calculation of $\delta\lambda$.
- ii Similarly, you may set $m = 0$ in the self-energy when you are calculating δZ .
- iii Also, when computing the divergent part of a graph to obtain $\delta\lambda$ take advantage of the fact that the overall divergence that you need is independent of the external momenta, and set some appropriate momenta to zero.

11. The full Lagrangian with counterterms for QED with a photon mass term is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2 - \frac{\lambda}{2}(\partial \cdot A)^2 + i\bar{\psi}\not{\partial}\psi + \mu^{\epsilon/2}e\bar{\psi}\not{A}\psi - M\bar{\psi}\psi \\ &\quad - \frac{Z_3 - 1}{4}F_{\mu\nu}^2 + (Z_2 - 1)i\bar{\psi}\not{\partial}\psi + (Z_2 - 1)\mu^{\epsilon/2}e\bar{\psi}\not{A}\psi - (M_0Z_2 - 1)\bar{\psi}\psi.\end{aligned}\quad (124)$$

Note that there are no counterterms for the photon mass and gauge fixing terms. This can be proved.

- (a) What is the renormalization group equation for the Green functions of the theory?
- (b) From the renormalization counterterms (see [2]), derive the renormalization group coefficients (β , etc).

Note that the bare coupling and fields are defined by

$$e_0 = \mu^{\epsilon/2}eZ_3^{-1/2}, \quad A_0 = AZ_3^{1/2}, \quad \psi_0 = \psi Z_2^{1/2}, \quad (125)$$

with corresponding definitions for the bare mass of the photon and the bare gauge fixing parameter. Note also that the total variation with respect to μ is

$$\mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \mu \frac{de}{d\mu} \frac{\partial}{\partial e} + \mu \frac{dM}{d\mu} \frac{\partial}{\partial M} + \mu \frac{dm^2}{d\mu} \frac{\partial}{\partial m^2} + \mu \frac{d\lambda}{d\mu} \frac{\partial}{\partial \lambda}. \quad (126)$$

12. In the theory of a combined ϕ^3 and ϕ^4 interaction

$$\mathcal{L} = \frac{1}{2}\partial\phi^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4, \quad (127)$$

find all the UV divergent one-loop graphs. Which, if any, counterterms are needed beyond those for terms in Eq. (127)? Calculate the one-loop counterterm for the three-point coupling g , using minimal subtraction for the renormalization prescription.

13. In this and the next problem, you will find the coordinate-space propagator for a free scalar field of mass m in an n -dimensional space-time. The propagator is

$$i\Delta_F(x; n, m) = \int \frac{d^n q}{(2\pi)^n} \frac{ie^{iq \cdot x}}{q^2 - m^2 + i\epsilon}, \quad (128)$$

and it satisfies the equation

$$(\square + m^2)\Delta_F = -\delta^{(n)}(x). \quad (129)$$

(a) Now, Δ_F is a function of x^2 alone. Write

$$\Delta_F = (-x^2)^{1/2-n/4} w(m\sqrt{-x^2}), \quad (130)$$

and show that w satisfies the modified Bessel equation of order $\nu = n/2 - 1$, when $x^2 \neq 0$. (You'll need to look up the properties of such functions. Standard solutions are named $I_{\pm\nu}$ and K_ν . The most general solution of the equation for w is a linear combination of two standard solutions of the modified Bessel equation. So some boundary conditions are needed. One boundary condition is that $\Delta_F \rightarrow 0$ as $-x^2 \rightarrow \infty$, and the other is that the δ -function in Eq. (129) be obtained.)

(b) To obtain the correct boundary condition at $x = 0$, Wick-rotate time (x^0) to be imaginary, integrate Eq. (129) over the interior of a small hypersphere centered at the origin, and use the divergence theorem. Show that the leading singularity of Δ_F as $x \rightarrow 0$ is

$$\Delta_F \sim \frac{-i\Gamma(n/2 - 1)}{4\pi^{n/2} (-x^2)^{n/2-1}}. \quad (131)$$

(c) Now apply the two boundary conditions to find a formula for Δ_F in terms of standard Bessel functions, Eq. (24).

14. In Minkowski space, the propagator $i\Delta_F(x^2)$ is singular, not only at $x = 0$, but on the whole light-cone $x^2 = 0$. Given the $i\epsilon$ in the momentum-space

propagator, as in Eq. (128), what is the correct $i\epsilon$ prescription in coordinate space? You may find the book by Bogoliubov and Shirkov[4] helpful for this and the previous problem.

References

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- [10] This and other relevant formulae and references can be found in: “Review of Particle Properties”, *Phys. Rev. D* **50**, 1173 (1994), and in “Handbook of Perturbative QCD”, *Rev. Mod. Phys.* **67**, 157 (1995).